

# Finiteness Principles for Smooth Selection

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## Introduction

In this paper and [18], we extend a basic finiteness principle [6, 10], used in [15, 16] to fit smooth functions  $F$  to data. Our results raise the hope that one can start to understand constrained interpolation problems in which e.g. the interpolating function  $F$  is required to be nonnegative.

Let us set up notation. We fix positive integers  $m, n, D$ . We will work with the function spaces  $C^m(\mathbb{R}^n, \mathbb{R}^D)$  and  $C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$  and their norms  $\|F\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)}$  and  $\|F\|_{C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)}$ . Here,  $C^m(\mathbb{R}^n, \mathbb{R}^D)$  denotes the space of all functions  $F : \mathbb{R}^n \rightarrow \mathbb{R}^D$  whose derivatives  $\partial^\beta F$  (for all  $|\beta| \leq m$ ) are continuous and bounded on  $\mathbb{R}^n$ , and  $C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$  denotes the space of all  $F : \mathbb{R}^n \rightarrow \mathbb{R}^D$  whose derivatives  $\partial^\beta F$  (for all  $|\beta| \leq m-1$ ) are bounded and Lipschitz on  $\mathbb{R}^n$ . When  $D = 1$ , we write  $C^m(\mathbb{R}^n)$  and  $C^{m-1,1}(\mathbb{R}^n)$  in place of  $C^m(\mathbb{R}^n, \mathbb{R}^D)$  and  $C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$ .

Expressions  $c(m, n)$ ,  $C(m, n)$ ,  $k(m, n)$ , etc. denote constants depending only on  $m, n$ ; these expressions may denote different constants in different occurrences. Similar conventions apply to constants denoted by  $C(m, n, D)$ ,  $k(D)$ , etc.

If  $X$  is any finite set, then  $\#(X)$  denotes the number of elements in  $X$ .

We recall the basic finiteness principle of [10].

**Theorem 1** *For large enough  $k^\# = k(m, n)$  and  $C^\# = C(m, n)$  the following hold:*

- (A)  **$C^m$  FLAVOR** *Let  $f : E \rightarrow \mathbb{R}$  with  $E \subset \mathbb{R}^n$  finite. Suppose that for each  $S \subset E$  with  $\#(S) \leq k^\#$  there exists  $F^S \in C^m(\mathbb{R}^n)$  with norm  $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$ , such that  $F^S = f$  on  $S$ . Then there exists  $F \in C^m(\mathbb{R}^n)$  with norm  $\|F\|_{C^m(\mathbb{R}^n)} \leq C^\#$ , such that  $F = f$  on  $E$ .*

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(B)  $C^{m-1,1}$  **FLAVOR** Let  $f : E \rightarrow \mathbb{R}$  with  $E \subset \mathbb{R}^n$  arbitrary. Suppose that for each  $S \subset E$  with  $\#(S) \leq k^\#$ , there exists  $F^S \in C^{m-1,1}(\mathbb{R}^n)$  with norm  $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq 1$ , such that  $F^S = f$  on  $S$ . Then there exists  $F \in C^{m-1,1}(\mathbb{R}^n)$  with norm  $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C^\#$ , such that  $F = f$  on  $E$ .

Theorem 1 and several related results were conjectured by Y. Brudnyi and P. Shvartsman in [5] and [6] (see also [23–25]). The first nontrivial case  $m = 2$  with the sharp “finiteness constant”  $k^\# = 3 \cdot 2^{n-1}$  was proven by P. Shvartsman [22, 25]; see also [9, 23]. The proof of Theorem 1 for general  $m, n$  appears in [10]. For general  $m, n$ , the optimal  $k^\#$  is unknown, but see [1, 30].

The proof [23, 25] of Theorem 1 for  $m = 2$  was based on a generalization of the following “finiteness principle for Lipschitz selection” [24] for maps of metric spaces.

**Theorem 2** For large enough  $k^\# = k^\#(D)$  and  $C^\# = C^\#(D)$ , the following holds.

Let  $X$  be a metric space. For each  $x \in X$ , let  $K(x) \subset \mathbb{R}^D$  be an affine subspace in  $\mathbb{R}^D$  of dimension  $\leq d$ . Suppose that for each  $S \subset X$  with  $\#(S) \leq k^\#$  there exists a map  $F^S : S \rightarrow \mathbb{R}^D$  with Lipschitz constant  $\leq 1$ , such that  $F^S(x) \in K(x)$  for all  $x \in S$ .

Then there exists a map  $F : X \rightarrow \mathbb{R}^D$  with Lipschitz constant  $\leq C^\#$ , such that  $F(x) \in K(x)$  for all  $x \in X$ .

In fact, P. Shvartsman in [24] showed that one can take  $k^\# = 2^{d+1}$  in Theorem 2 and that the constant  $k^\# = 2^{d+1}$  is sharp, see [26].

P. Shvartsman also showed that Theorem 2 remains valid when  $\mathbb{R}^D$  is replaced by a Hilbert space (see [27]) or a Banach space (see [29]).

It is conjectured in [6] that Theorem 2 should hold for any compact convex subsets  $K(x) \subset \mathbb{R}^D$ . In [28], P. Shvartsman provided evidence for this conjecture: He showed that the conjecture holds in the case when  $D = 2$  and in the case when  $X$  is a finite metric space and the constant  $C^\#$  is allowed to depend on the cardinality of  $X$ .

In this paper we prove finiteness principles for  $C^m(\mathbb{R}^n, \mathbb{R}^D)$ -selection, and for  $C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$ -selection, in particular providing a proof for the conjecture in [6] for the case  $X = \mathbb{R}^n$ .

**Theorem 3** For large enough  $k^\# = k(m, n, D)$  and  $C^\# = C(m, n, D)$ , the following hold.

(A)  $C^m$  **FLAVOR** Let  $E \subset \mathbb{R}^n$  be finite. For each  $x \in E$ , let  $K(x) \subset \mathbb{R}^D$  be convex. Suppose that for each  $S \subset E$  with  $\#(S) \leq k^\#$ , there exists  $F^S \in C^m(\mathbb{R}^n, \mathbb{R}^D)$  with norm  $\|F^S\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq 1$ , such that  $F^S(x) \in K(x)$  for all  $x \in S$ .

Then there exists  $F \in C^m(\mathbb{R}^n, \mathbb{R}^D)$  with norm  $\|F\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq C^\#$ , such that  $F(x) \in K(x)$  for all  $x \in E$ .

- (B)  $C^{m-1,1}$  **FLAVOR** Let  $E \subset \mathbb{R}^n$  be arbitrary. For each  $x \in E$ , let  $K(x) \subset \mathbb{R}^n$  be closed and convex. Suppose that for each  $S \subset E$  with  $\#(S) \leq k^\#$ , there exists  $F^S \in C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$  with norm  $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)} \leq 1$ , such that  $F^S(x) \in K(x)$  for all  $x \in S$ . Then there exists  $F \in C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$  with norm  $\|F\|_{C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)} \leq C^\#$ , such that  $F(x) \in K(x)$  for all  $x \in E$ .

In a forthcoming paper [18], we will prove the following closely related result on interpolation by nonnegative functions.

**Theorem 4** For large enough  $k^\# = k(m, n)$  and  $C^\# = C(m, n)$  the following hold.

- (A)  $C^m$  **FLAVOR** Let  $f : E \rightarrow [0, \infty)$  with  $E \subset \mathbb{R}^n$  finite. Suppose that for each  $S \subset E$  with  $\#(S) \leq k^\#$ , there exists  $F^S \in C^m(\mathbb{R}^n)$  with norm  $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$ , such that  $F^S = f$  on  $S$  and  $F^S \geq 0$  on  $\mathbb{R}^n$ . Then there exists  $F \in C^m(\mathbb{R}^n)$  with norm  $\|F\|_{C^m(\mathbb{R}^n)} \leq C^\#$ , such that  $F = f$  on  $E$  and  $F \geq 0$  on  $\mathbb{R}^n$ .
- (B)  $C^{m-1,1}$  **FLAVOR** Let  $f : E \rightarrow [0, \infty)$  with  $E \subset \mathbb{R}^n$  arbitrary. Suppose that for each  $S \subset E$  with  $\#(S) \leq k^\#$ , there exists  $F^S \in C^{m-1,1}(\mathbb{R}^n)$  with norm  $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq 1$ , such that  $F^S = f$  on  $S$  and  $F^S \geq 0$  on  $\mathbb{R}^n$ . Then there exists  $F \in C^{m-1,1}(\mathbb{R}^n)$  with norm  $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C^\#$ , such that  $F = f$  on  $E$  and  $F \geq 0$  on  $\mathbb{R}^n$ .

One might ask how to decide whether there exist  $F^S$  as in the above results. For Theorem 1 this issue is addressed in [10]; for Theorems 3 and 4 we address it in this paper and [18].

A weaker version of the case  $D = 1$  of Theorem 3 appears in [10]. There, each  $K(x)$  is an interval  $[f(x) - \varepsilon(x), f(x) + \varepsilon(x)]$ . In place of the conclusion  $F(x) \in K(x)$  in Theorem 3, [10] obtains the weaker conclusion  $F(x) \in [f(x) - C\varepsilon(x), f(x) + C\varepsilon(x)]$  for a constant  $C$  determined by  $m, n$ .

Our interest in Theorems 3 and 4 arises in part from their possible connection to the interpolation algorithms of Fefferman-Klartag [15, 16]. Given a function  $f : E \rightarrow \mathbb{R}$  with  $E \subset \mathbb{R}^n$  finite, the goal of [15, 16] is to compute a function  $F \in C^m(\mathbb{R}^n)$  such that  $F = f$  on  $E$ , with norm  $\|F\|_{C^m(\mathbb{R}^n)}$  as small as possible up to a factor  $C(m, n)$ . Roughly speaking, the algorithm in [15, 16] computes such an  $F$  using  $O(N \log N)$  computer operations, where  $N = \#(E)$ . The algorithm is based on ideas from the proof [10] of Theorem 1. Accordingly, Theorems 3 and 4 raise the hope that we can start to understand constrained interpolation problems, in which e.g. the interpolant  $F$  is constrained to be nonnegative everywhere on  $\mathbb{R}^n$ .

Theorems 3 and 4 follow from a more general result on  $C^m$  functions whose Taylor polynomials belong to prescribed convex sets. More precisely, let  $J_x(F)$  denote the  $(m-1)^{\text{st}}$  degree Taylor polynomial of  $F$  at  $x$ ; thus,  $J_x(F)$  belongs to the vector space of  $\mathcal{P}$  of all such polynomials. Let  $E \subset \mathbb{R}^n$  be finite. For

each  $x \in E$  and  $M > 0$ , let  $\Gamma(x, M)$  be a convex subset of  $\mathcal{P}$ . Under suitable hypotheses on the  $\Gamma(x, M)$  we will prove the following:

**Finiteness Principle** *Fix  $M_0 > 0$ . Suppose that for each  $S \subseteq E$  with  $\#(S) \leq k^\#(m, n)$  there exists  $F^S \in C^m(\mathbb{R}^n)$  with  $\|F^S\|_{C^m(\mathbb{R}^n)} \leq M_0$ , such that  $J_x(F^S) \in \Gamma(x, M_0)$  for all  $x \in S$ .*

*Then there exists  $F \in C^m(\mathbb{R}^n)$  with  $\|F\|_{C^m(\mathbb{R}^n)} \leq CM_0$ , such that  $J_x(F) \in \Gamma(x, CM_0)$  for all  $x \in E$ .*

*Here,  $C$  depends only on  $m, n$  and the constants in our assumptions on the  $\Gamma(x, M)$ .*

See Section III.1 below for our assumptions on the  $\Gamma(x, M)$ . The precise statement of the above finiteness principle is given by Theorem 6 in Section III.1 and the remark after its proof. That result and the closely related Theorem 5 in Section II.10 form the real content of this paper.

A special case of Theorem 6 was proven in [11]. In that special case, the sets  $\Gamma(x, M)$  have the form  $\Gamma(x, M) = f_x + M \cdot \sigma(x)$  with  $f_x \in \mathcal{P}$  and  $\sigma(x)$  a symmetric convex subset of  $\mathcal{P}$ . The  $\sigma(x)$  are required to be “Whitney convex”; see [11].

Here, to prove Theorems 5 and 6, we adapt the arguments in [11] to more general families of convex sets  $\Gamma(x, M)$ . In particular, our hypothesis of “ $(C_w, \delta_{\max})$ -convexity”, formulated in Section I.2 below, generalizes the notion of Whitney convexity to the present context.

However, at one crucial point (Case 2 in the proof of Lemma 15 below) the argument has no analogue in [11].

We refer the reader to the expository paper [14] for an explanation of the main ideas in [11].

This paper is part of a literature on extension, interpolation, and selection of functions, going back to H. Whitney’s seminal work [32], and including fundamental contributions by G. Glaeser [19], Y. Brudnyi and P. Shvartsman [4, 6–9, 22–30], J. Wells [31], E. Le Gruyer [20], and E. Bierstone, P. Milman, and W. Pawłucki [1–3], as well as our own papers [10–17]. See e.g. [14] for the history of the problem, as well as Zobin [33, 34] for a related problem.

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# Part I

## Shape Fields and Their Refinements

### I.1 Notation and Preliminaries

Fix  $m, n \geq 1$ . We will work with cubes in  $\mathbb{R}^n$ ; all our cubes have sides parallel to the coordinate axes. If  $Q$  is a cube, then  $\delta_Q$  denotes the sidelength of  $Q$ . For real numbers  $A > 0$ ,  $AQ$  denotes the cube whose center is that of  $Q$ , and whose sidelength is  $A\delta_Q$ .

A dyadic cube is a cube of the form  $I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$ , where each  $I_\nu$  has the form  $[2^k \cdot i_\nu, 2^k \cdot (i_\nu + 1))$  for integers  $i_1, \dots, i_n, k$ . Each dyadic cube  $Q$  is contained in one and only one dyadic cube with sidelength  $2\delta_Q$ ; that cube is denoted by  $Q^+$ .

We write  $\mathcal{P}$  to denote the vector space of all real-valued polynomials of degree at most  $(m-1)$  on  $\mathbb{R}^n$ . If  $x \in \mathbb{R}^n$  and  $F$  is a real-valued  $C^{m-1}$  function on a neighborhood of  $x$ , then  $J_x(F)$  (the “jet” of  $F$  at  $x$ ) denotes the  $(m-1)^{\text{st}}$  order Taylor polynomial of  $F$  at  $x$ . Thus,  $J_x(F) \in \mathcal{P}$ .

For each  $x \in \mathbb{R}^n$ , there is a natural multiplication  $\odot_x$  on  $\mathcal{P}$  (“multiplication of jets at  $x$ ”) defined by setting

$$P \odot_x Q = J_x(PQ) \text{ for } P, Q \in \mathcal{P}.$$

We write  $C^m(\mathbb{R}^n)$  to denote the Banach space of real-valued  $C^m$  functions  $F$  on  $\mathbb{R}^n$  for which the norm

$$\|F\|_{C^m(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \leq m} |\partial^\alpha F(x)|$$

is finite. For  $D \geq 1$ , we write  $C^m(\mathbb{R}^n, \mathbb{R}^D)$  to denote the Banach space of all  $\mathbb{R}^D$ -valued  $C^m$  functions  $F$  on  $\mathbb{R}^n$ , for which the norm

$$\|F\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} = \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \leq m} \|\partial^\alpha F(x)\|$$

is finite. Here, we use the Euclidean norm on  $\mathbb{R}^D$ .

If  $F$  is a real-valued function on a cube  $Q$ , then we write  $F \in C^m(Q)$  to denote that  $F$  and its derivatives up to  $m$ -th order extend continuously to the closure of  $Q$ . For  $F \in C^m(Q)$ , we define

$$\|F\|_{C^m(Q)} = \sup_{x \in Q} \max_{|\alpha| \leq m} |\partial^\alpha F(x)|.$$

Similarly, if  $F$  is an  $\mathbb{R}^D$ -valued function on a cube  $Q$ , then we write  $F \in C^m(Q, \mathbb{R}^D)$  to denote that  $F$  and its derivatives up to  $m$ -th order extend con-

tinuously to the closure of  $Q$ . For  $F \in C^m(Q, \mathbb{R}^D)$ , we define

$$\|F\|_{C^m(Q, \mathbb{R}^D)} = \sup_{x \in Q} \max_{|\alpha| \leq m} \|\partial^\alpha F(x)\|,$$

where again we use the Euclidean norm on  $\mathbb{R}^D$ .

If  $F \in C^m(Q)$  and  $x$  belongs to the boundary of  $Q$ , then we still write  $J_x(F)$  to denote the  $(m-1)^{\text{rst}}$  degree Taylor polynomial of  $F$  at  $x$ , even though  $F$  isn't defined on a full neighborhood of  $x \in \mathbb{R}^n$ .

Let  $S \subset \mathbb{R}^n$  be non-empty and finite. A Whitney field on  $S$  is a family of polynomials

$$\vec{P} = (P^y)_{y \in S} \quad (\text{each } P^y \in \mathcal{P}),$$

parametrized by the points of  $S$ .

We write  $\text{Wh}(S)$  to denote the vector space of all Whitney fields on  $S$ .

For  $\vec{P} = (P^y)_{y \in S} \in \text{Wh}(S)$ , we define the seminorm

$$\|\vec{P}\|_{\dot{C}^m(S)} = \max_{x, y \in S, (x \neq y), |\alpha| \leq m} \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x - y|^{m-|\alpha|}}.$$

(If  $S$  consists of a single point, then  $\|\vec{P}\|_{\dot{C}^m(S)} = 0$ .)

We write  $\mathcal{M}$  to denote the set of all multiindices  $\alpha = (\alpha_1, \dots, \alpha_n)$  of order  $|\alpha| = \alpha_1 + \dots + \alpha_n \leq m-1$ .

We define a (total) order relation  $<$  on  $\mathcal{M}$ , as follows. Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$  be distinct elements of  $\mathcal{M}$ . Pick the largest  $k$  for which  $\alpha_1 + \dots + \alpha_k \neq \beta_1 + \dots + \beta_k$ . (There must be at least one such  $k$ , since  $\alpha$  and  $\beta$  are distinct). Then we say that  $\alpha < \beta$  if  $\alpha_1 + \dots + \alpha_k < \beta_1 + \dots + \beta_k$ .

We also define a (total) order relation  $<$  on subsets of  $\mathcal{M}$ , as follows. Let  $\mathcal{A}, \mathcal{B}$  be distinct subsets of  $\mathcal{M}$ , and let  $\gamma$  be the least element of the symmetric difference  $(\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A})$  (under the above order on the elements of  $\mathcal{M}$ ). Then we say that  $\mathcal{A} < \mathcal{B}$  if  $\gamma \in \mathcal{A}$ .

One checks easily that the above relations  $<$  are indeed total order relations. Note that  $\mathcal{M}$  is minimal, and the empty set  $\emptyset$  is maximal under  $<$ . A set  $\mathcal{A} \subseteq \mathcal{M}$  is called monotonic if, for all  $\alpha \in \mathcal{A}$  and  $\gamma \in \mathcal{M}$ ,  $\alpha + \gamma \in \mathcal{M}$  implies  $\alpha + \gamma \in \mathcal{A}$ . We make repeated use of a simple observation:

Suppose  $\mathcal{A} \subseteq \mathcal{M}$  is monotonic,  $P \in \mathcal{P}$  and  $x_0 \in \mathbb{R}^n$ . If  $\partial^\alpha P(x_0) = 0$  for all  $\alpha \in \mathcal{A}$ , then  $\partial^\alpha P \equiv 0$  on  $\mathbb{R}^n$  for  $\alpha \in \mathcal{A}$ .

This follows by writing  $\partial^\alpha P(y) = \sum_{|\gamma| \leq m-1-|\alpha|} \frac{1}{\gamma!} \partial^{\alpha+\gamma} P(x_0) \cdot (y - x_0)^\gamma$  and noting that all the relevant  $\alpha + \gamma$  belong to  $\mathcal{A}$ , hence  $\partial^{\alpha+\gamma} P(x_0) = 0$ .

We need a few elementary facts about convex sets. We recall

**Helly's Theorem** *Let  $K_1, \dots, K_N \subset \mathbb{R}^D$  be convex. Suppose that  $K_{i_1} \cap \dots \cap K_{i_{D+1}}$  is nonempty for any  $i_1, \dots, i_{D+1} \in \{1, \dots, N\}$ . Then  $K_1 \cap \dots \cap K_N$  is nonempty.*

See [21].

We also use the following

**Trivial Remark on Convex Sets** *Let  $\Gamma$  be a convex set, and let  $P_0, P_0 + P_v, P_0 - P_v \in \Gamma$  for  $v = 1, \dots, v_{\max}$ . Then for any real numbers  $t_1, \dots, t_{v_{\max}}$  with*

$$\sum_{v=1}^{v_{\max}} |t_v| \leq 1,$$

*we have*

$$P_0 + \sum_{v=1}^{v_{\max}} t_v P_v \in \Gamma.$$

If  $\lambda = (\lambda_1, \dots, \lambda_n)$  is an  $n$ -tuple of positive real numbers, and if  $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ , then we write  $\lambda^\beta$  to denote

$$\lambda_1^{\beta_1} \dots \lambda_n^{\beta_n}.$$

We write  $B_n(x, r)$  to denote the open ball in  $\mathbb{R}^n$  with center  $x$  and radius  $r$ , with respect to the Euclidean metric.

## I.2 Shape Fields

Let  $E \subset \mathbb{R}^n$  be finite. For each  $x \in E$ ,  $M \in (0, \infty)$ , let  $\Gamma(x, M) \subseteq \mathcal{P}$  be a (possibly empty) convex set. We say that  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  is a shape field if for all  $x \in E$  and  $0 < M' \leq M < \infty$ , we have

$$\Gamma(x, M') \subseteq \Gamma(x, M).$$

Let  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  be a shape field and let  $C_w, \delta_{\max}$  be positive real numbers. We say that  $\vec{\Gamma}$  is  $(C_w, \delta_{\max})$ -convex if the following condition holds:

Let  $0 < \delta \leq \delta_{\max}$ ,  $x \in E$ ,  $M \in (0, \infty)$ ,  $P_1, P_2, Q_1, Q_2 \in \mathcal{P}$ . Assume that

- (1)  $P_1, P_2 \in \Gamma(x, M)$ ;
- (2)  $|\partial^\beta(P_1 - P_2)(x)| \leq M\delta^{m-|\beta|}$  for  $|\beta| \leq m-1$ ;
- (3)  $|\partial^\beta Q_i(x)| \leq \delta^{-|\beta|}$  for  $|\beta| \leq m-1$  for  $i = 1, 2$ ;
- (4)  $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$ .

Then

- (5)  $P := Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2 \in \Gamma(x, C_w M)$ .

The following lemma follows easily from the definition of  $(C_w, \delta_{\max})$ -convexity.

**Lemma 1** Suppose  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  is a  $(C_w, \delta_{\max})$ -convex shape field. Let

$$(6) \quad 0 < \delta \leq \delta_{\max}, \quad x \in E, \quad M > 0, \quad P_1, P_2, Q_1, Q_2 \in \mathcal{P} \text{ and } A', A'' > 0.$$

Assume that

$$(7) \quad P_1, P_2 \in \Gamma(x, A'M);$$

$$(8) \quad |\partial^\beta (P_1 - P_2)(x)| \leq A'M\delta^{m-|\beta|} \text{ for } |\beta| \leq m-1;$$

$$(9) \quad |\partial^\beta Q_i(x)| \leq A''\delta^{-|\beta|} \text{ for } |\beta| \leq m-1 \text{ and } i = 1, 2;$$

$$(10) \quad Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1.$$

Then

$$(11) \quad P := Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2 \in \Gamma(x, CM) \text{ with } C \text{ determined by } A', A'', C_w, m, \text{ and } n.$$

By Lemma 1 and an induction argument, we also have the following result.

**Lemma 2** Suppose  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  is a  $(C_w, \delta_{\max})$ -convex shape field. Let

$$(12) \quad 0 < \delta \leq \delta_{\max}, \quad x \in E, \quad M > 0, \quad A', A'' > 0, \quad P_1, \dots, P_k, Q_1, \dots, Q_k \in \mathcal{P}.$$

Assume that

$$(13) \quad P_i \in \Gamma(x, A'M) \text{ for } i = 1, \dots, k;$$

$$(14) \quad |\partial^\beta (P_i - P_j)(x)| \leq A'M\delta^{m-|\beta|} \text{ for } |\beta| \leq m-1, \quad i, j = 1, \dots, k;$$

$$(15) \quad |\partial^\beta Q_i(x)| \leq A''\delta^{-|\beta|} \text{ for } |\beta| \leq m-1 \text{ and } i = 1, \dots, k;$$

$$(16) \quad \sum_{i=1}^k Q_i \odot_x Q_i = 1.$$

Then

$$(17) \quad \sum_{i=1}^k Q_i \odot_x Q_i \odot_x P_i \in \Gamma(x, CM), \text{ with } C \text{ determined by } A', A'', C_w, m, n, k.$$

Next, we define the first refinement of a shape field  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  to be  $\vec{\Gamma}^\# = (\Gamma^\#(x, M))_{x \in E, M > 0}$ , where  $\Gamma^\#(x, M)$  consists of those  $P^\# \in \mathcal{P}$  such that for all  $y \in E$  there exists  $P \in \Gamma(y, M)$  for which

$$(18) \quad |\partial^\beta (P^\# - P)(x)| \leq M|x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Note that each  $\Gamma^\#(x, M)$  is a (possibly empty) convex subset of  $\mathcal{P}$ , and that  $M' \leq M$  implies  $\Gamma^\#(x, M') \subseteq \Gamma^\#(x, M)$ . Thus,  $\vec{\Gamma}^\#$  is again a shape field. Taking  $y = x$  in (18), we see that  $\Gamma^\#(x, M) \subset \Gamma(x, M)$ .



**Lemma 3** *Let  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field, and let  $\vec{\Gamma}^\# = (\Gamma^\#(x, M))_{x \in E, M > 0}$  be the first refinement of  $\vec{\Gamma}^\#$ . Then  $\vec{\Gamma}^\#$  is  $(C, \delta_{\max})$ -convex, where  $C$  is determined by  $C_w, m, n$ .*

**Proof.** We write  $c, C, C'$ , etc., to denote constants determined by  $C_w, m, n$ . These symbols may denote different constants in different occurrences. Let  $0 < \delta \leq \delta_{\max}, M > 0, x \in E, P_1^\#, P_2^\#, Q_1^\#, Q_2^\# \in \mathcal{P}$ , and assume that

$$(19) \quad P_i^\# \in \Gamma^\#(x, M) \text{ for } i = 1, 2;$$

$$(20) \quad \left| \partial^\beta \left( P_1^\# - P_2^\# \right) (x) \right| \leq M \delta^{m-|\beta|} \text{ for } |\beta| \leq m-1;$$

$$(21) \quad \left| \partial^\beta Q_i^\#(x) \right| \leq \delta^{-|\beta|} \text{ for } |\beta| \leq m-1, i = 1, 2; \text{ and}$$

$$(22) \quad Q_1^\# \odot_x Q_1^\# + Q_2^\# \odot_x Q_2^\# = 1.$$

Under the above assumptions, we must show that

$$(23) \quad P^\# := Q_1^\# \odot_x Q_1^\# \odot_x P_1^\# + Q_2^\# \odot_x Q_2^\# \odot_x P_2^\# \in \Gamma^\#(x, CM).$$

By definition of  $\Gamma^\#(\cdot, \cdot)$ , this means that given any  $y \in E$  there exists

$$(24) \quad P \in \Gamma(y, CM) \text{ such that}$$

$$(25) \quad |\partial^\beta (P^\# - P)(x)| \leq CM |x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1, \text{ where}$$

Thus, to prove Lemma 3, we must prove that there exists  $P$  satisfying (24), (25), under the assumptions (19)–(22). To do so, we start by defining the functions

$$(26) \quad \theta_i = \frac{Q_i^\#}{[(Q_1^\#)^2 + (Q_2^\#)^2]^{1/2}} \text{ on } B_n(x, c_0 \delta) \text{ (} i = 1, 2 \text{)}.$$

We pick  $c_0 < 1$  small enough so that (21), (22) guarantee that  $\theta_i$  is well-defined on  $B_n(x, c_0 \delta)$  and satisfies

$$(27) \quad |\partial^\beta \theta_i| \leq C \delta^{-|\beta|} \text{ on } B_n(x, c_0 \delta) \text{ for } |\beta| \leq m, i = 1, 2,$$

and

$$(28) \quad \theta_1^2 + \theta_2^2 = 1 \text{ on } B_n(x, c_0 \delta).$$

Also

$$(29) \quad J_x(\theta_i) = Q_i^\# \text{ for } i = 1, 2,$$

thanks to (22).

We now divide the discussion of (24), (25) into two cases.

CASE 1: Suppose  $y \in B_n(x, c_0 \delta)$ .

By (19) and the definition of  $\Gamma^\#(\cdot, \cdot)$ , there exist

$$(30) \quad P_i \in \Gamma(y, M) \quad (i = 1, 2)$$

satisfying

$$(31) \quad |\partial^\beta (P_i^\# - P_i)(x)| \leq M|x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1, i = 1, 2.$$

Since we are in CASE 1, estimates (20) and (31) together imply that

$$(32) \quad |\partial^\beta (P_1 - P_2)(x)| \leq CM\delta^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Consequently,

$$(33) \quad |\partial^\beta (P_1 - P_2)| \leq CM\delta^{m-|\beta|} \text{ on } B_n(x, c_0\delta) \text{ for } |\beta| \leq m.$$

(Recall that  $P_1, P_2$  are polynomials of degree at most  $m-1$ .)

Thanks to (27), (28), (30), (33), and the  $(C_w, \delta_{\max})$ -convexity of  $\vec{\Gamma}$ , we may apply Lemma 1 to the polynomials  $P_1, P_2, Q_1, Q_2$ , where  $Q_i = J_y(\theta_i)$  for  $i = 1, 2$ . This tells us that

$$(34) \quad P := J_y(\theta_1^2 P_1 + \theta_2^2 P_2) \in \Gamma(y, CM).$$

That is, the  $P$  in (34) satisfies (24). We will show that it also satisfies (25). Thanks to (23), (29), we have

$$(35) \quad P^\# = J_x(\theta_1^2 P_1^\# + \theta_2^2 P_2^\#).$$

In view of (34), (35), our desired estimate (25) is equivalent to the following:

$$(36) \quad \left| \partial^\beta \left( \theta_1^2 P_1^\# + \theta_2^2 P_2^\# - J_y \left( \theta_1^2 P_1 + \theta_2^2 P_2 \right) \right) (x) \right| \leq CM|x - y|^{m-|\beta|}$$

for  $|\beta| \leq m-1$ .

Thus, we have reduced the existence of  $P$  satisfying (24), (25) in CASE 1 to the task of proving (36).

Since  $\theta_1^2 + \theta_2^2 = 1$  (see (28)) and  $J_y P_1 = P_1$ , the following holds on  $B_n(x, c_0\delta)$ :

$$\begin{aligned} & \left( \theta_1^2 P_1^\# + \theta_2^2 P_2^\# - J_y \left( \theta_1^2 P_1 + \theta_2^2 P_2 \right) \right) \\ &= \theta_1^2 (P_1^\# - P_1) + \theta_2^2 (P_2^\# - P_2) + \left[ \theta_2^2 (P_2 - P_1) - J_y \left( \theta_2^2 (P_2 - P_1) \right) \right]. \end{aligned}$$

Consequently, the desired estimate (36) will follow if we can show that

$$(37) \quad \left| \partial^\beta \left[ \theta_i^2 (P_i^\# - P_i) \right] (x) \right| \leq CM|x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1, i = 1, 2,$$

and

$$(38) \quad \left| \partial^\beta \left[ \theta_2^2 (P_1 - P_2) - J_y \left( \theta_2^2 (P_1 - P_2) \right) \right] (x) \right| \leq CM|x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Moreover, (37) follows at once from (27) and (31), since  $\delta^{-|\beta|} \leq C|x-y|^{-|\beta|}$  in CASE 1.

To check (38), we apply (27) and (33) to deduce that

$$\left| \partial^\beta \left[ \theta_2^2 (P_1 - P_2) \right] \right| \leq CM$$

on  $B(x, c_0\delta)$  for  $|\beta| = m$ .

Therefore, (38) follows from Taylor's theorem.

This proves the existence of a  $P$  satisfying (24), (25) in CASE 1.

CASE 2: Suppose that  $y \notin B_n(x, c_0\delta)$ .

Since  $P_1^\# \in \Gamma(x, M)$  (see (19)), there exists

$$(39) \quad P_1 \in \Gamma(y, M)$$

such that

$$(40) \quad \left| \partial^\beta \left( P_1^\# - P_1 \right) (x) \right| \leq M|x-y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Thanks to (22), we may rewrite  $P^\#$  in the form

$$P^\# = P_1^\# + Q_2^\# \odot_x Q_2^\# \odot_x \left( P_2^\# - P_1^\# \right).$$

Our assumptions (20), (21) therefore yield the estimates

$$\left| \partial^\beta \left( P^\# - P_1^\# \right) (x) \right| \leq CM\delta^{m-|\beta|}$$

for  $|\beta| \leq m-1$ .

Since we are in CASE 2, it follows that

$$(41) \quad \left| \partial^\beta \left( P^\# - P_1^\# \right) (x) \right| \leq CM|x-y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

From (40) and (41), we learn that

$$(42) \quad \left| \partial^\beta \left( P^\# - P_1 \right) (x) \right| \leq CM|x-y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

We now know from (39) and (42) that  $P := P_1$  satisfies (24) and (25). Thus, in CASE 2 we again have a polynomial  $P$  satisfying (24) and (25). We have seen in all cases that there exists  $P \in \mathcal{P}$  satisfying (24), (25).

The proof of Lemma 3 is complete. ■

Next we define the higher refinements of a given shape field  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$ .

By induction on  $l \geq 0$ , we define  $\vec{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0}$ ; to do so, we start with our given  $\vec{\Gamma}_0$ , and define  $\vec{\Gamma}_{l+1}$  to be the first refinement of  $\vec{\Gamma}_l$ , for each  $l \geq 0$ . Thus, each  $\vec{\Gamma}_l$  is a shape field.

By the definition of the first refinement and Lemma 3, we have the following result.

**Lemma 4** (A) Let  $x, y \in E$ ,  $l \geq 1$ ,  $M > 0$ , and  $P \in \Gamma_l(x, M)$ . Then there exists  $P' \in \Gamma_{l-1}(y, M)$  such that

$$|\partial^\beta (P - P')(x)| \leq M |x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

(B) If  $\vec{\Gamma}_0$  is  $(C_w, \delta_{\max})$ -convex, then for each  $l \geq 0$ ,  $\vec{\Gamma}_l$  is  $(C_l, \delta_{\max})$ -convex, where  $C_l$  is determined by  $C_w, l, m, n$ .

We call  $\vec{\Gamma}_l$  the  $l$ -th refinement of  $\vec{\Gamma}_0$ . (This is consistent with our previous definition of the first refinement.)

### I.3 Polynomial Bases

Let  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  be a shape field. Let  $x_0 \in E$ ,  $M_0 > 0$ ,  $P^0 \in \mathcal{P}$ ,  $\mathcal{A} \subseteq \mathcal{M}$ ,  $P_\alpha \in \mathcal{P}$  for  $\alpha \in \mathcal{A}$ ,  $C_B > 0$ ,  $\delta > 0$  be given. Then we say that  $(P_\alpha)_{\alpha \in \mathcal{A}}$  is an  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$  if the following conditions are satisfied:

- (1)  $P^0 \in \Gamma(x_0, C_B M_0)$ .
- (2)  $P^0 + \frac{M_0 \delta^{m-|\alpha|}}{C_B} P_\alpha, P^0 - \frac{M_0 \delta^{m-|\alpha|}}{C_B} P_\alpha \in \Gamma(x_0, C_B M_0)$  for all  $\alpha \in \mathcal{A}$ .
- (3)  $\partial^\beta P_\alpha(x_0) = \delta_{\alpha\beta}$  (Kronecker delta) for  $\beta, \alpha \in \mathcal{A}$ .
- (4)  $|\partial^\beta P_\alpha(x_0)| \leq C_B \delta^{|\alpha|-|\beta|}$  for all  $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$ .

We say that  $(P_\alpha)_{\alpha \in \mathcal{A}}$  is a weak  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$  if conditions (1), (2), (3) hold as stated, and condition (4) holds for  $\alpha \in \mathcal{A}, \beta \in \mathcal{M}, \beta \geq \alpha$ .

We make a few obvious remarks.

- (5) Any  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$  is also an  $(\mathcal{A}, \delta, C'_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$ , whenever  $C'_B \geq C_B$ .
- (6) Any  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$  is also an  $(\mathcal{A}, \delta', C_B \cdot [\max\{\frac{\delta'}{\delta}, \frac{\delta}{\delta'}\}]^m)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$ , for any  $\delta' > 0$ .
- (7) Any weak  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$  is also a weak  $(\mathcal{A}, \delta', C'_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$ , whenever  $0 < \delta' \leq \delta$  and  $C'_B \geq C_B$ .

Note that (1) need not follow from (2), since  $\mathcal{A}$  may be empty.

- (8) If  $\mathcal{A} = \emptyset$ , then the existence of an  $(\mathcal{A}, \delta, C_B)$ -basis (or a weak  $(\mathcal{A}, \delta, C_B)$ -basis) for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$  is equivalent to the assertion that  $P^0 \in \Gamma(x_0, C_B M_0)$ .

The main result of this section is Lemma 7 below. The proof of Lemma 7 relies on two other lemmas.

As a consequence of Lemma 1, we have the following result.

**Lemma 5** *Let  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field. Fix  $x_0 \in E$ ,  $M_0 > 0$ ,  $0 < \delta \leq \delta_{\max}$ ,  $C_1 > 0$ , and let  $P^0, \hat{P}, \hat{S} \in \mathcal{P}$ .*

*Assume that*

$$(9) \quad P^0 + \frac{1}{C_1} \hat{P}, P^0 - \frac{1}{C_1} \hat{P} \in \Gamma(x_0, C_1 M_0);$$

$$(10) \quad |\partial^\beta \hat{P}(x_0)| \leq C_1 M_0 \delta^{m-|\beta|} \text{ for } |\beta| \leq m-1; \text{ and}$$

$$(11) \quad |\partial^\beta \hat{S}(x_0)| \leq C_1 \delta^{-|\beta|} \text{ for } |\beta| \leq m-1.$$

*Then*

$$(12) \quad P^0 + \frac{1}{C_2} \hat{S} \odot_{x_0} \hat{P}, P^0 - \frac{1}{C_2} \hat{S} \odot_{x_0} \hat{P} \in \Gamma(x_0, C_2 M_0), \text{ with } C_2 \text{ determined by } C_1, C_w, m, n.$$

We also need the following result, which is immediate<sup>1</sup> from Lemma 16.1 in [10].

**Lemma 6 (Rescaling Lemma)** *Let  $\mathcal{A} \subseteq \mathcal{M}$ , and let  $C, a$  be positive real numbers. Suppose we are given real numbers  $F_{\alpha, \beta}$ , indexed by  $\alpha \in \mathcal{A}$ ,  $\beta \in \mathcal{M}$ . Assume that the following conditions are satisfied.*

$$(13) \quad F_{\alpha, \alpha} \neq 0 \text{ for all } \alpha \in \mathcal{A}.$$

$$(14) \quad |F_{\alpha, \beta}| \leq C \cdot |F_{\alpha, \alpha}| \text{ for all } \alpha \in \mathcal{A}, \beta \in \mathcal{M} \text{ with } \beta \geq \alpha.$$

$$(15) \quad F_{\alpha, \beta} = 0 \text{ for all } \alpha, \beta \in \mathcal{A} \text{ with } \alpha \neq \beta.$$

*Then there exist positive numbers  $\lambda_1, \dots, \lambda_n$  and a map  $\phi : \mathcal{A} \rightarrow \mathcal{M}$ , with the following properties:*

$$(16) \quad c(a) \leq \lambda_i \leq 1 \text{ for each } i, \text{ where } c(a) \text{ is determined by } C, a, m, n;$$

$$(17) \quad \phi(\alpha) \leq \alpha \text{ for each } \alpha \in \mathcal{A};$$

$$(18) \quad \text{For each } \alpha \in \mathcal{A}, \text{ either } \phi(\alpha) = \alpha \text{ or } \phi(\alpha) \notin \mathcal{A}.$$

*Suppose we define  $\hat{F}_{\alpha, \beta} = \lambda^\beta F_{\alpha, \beta}$  for  $\alpha \in \mathcal{A}$ ,  $\beta \in \mathcal{M}$ , where we recall that  $\lambda^\beta$  denotes  $\lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n}$  for  $\beta = (\beta_1, \dots, \beta_n)$ . Then*

$$(19) \quad |\hat{F}_{\alpha, \beta}| \leq a \cdot |\hat{F}_{\alpha, \phi(\alpha)}| \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M} \setminus \{\phi(\alpha)\}.$$

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<sup>1</sup>Lemma 16.1 in [10] involves also real numbers  $F_{\alpha, \beta}$  with  $|\beta| = m$ . Setting those  $F_{\alpha, \beta} = 0$ , we recover the Rescaling Lemma stated here.

Lemma 3.3 in [10] tells us that any map  $\phi : \mathcal{A} \rightarrow \mathcal{M}$  satisfying (17), (18) satisfies also

$$(20) \quad \phi(\mathcal{A}) \leq \mathcal{A}, \text{ with equality only if } \phi \text{ is the identity map.}$$

We are ready to state the main result of this section.

**Lemma 7 (Relabeling Lemma)** *Let  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field. Let  $x_0 \in E$ ,  $M_0 > 0$ ,  $0 < \delta \leq \delta_{\max}$ ,  $C_B > 0$ ,  $P^0 \in \Gamma(x_0, M_0)$ ,  $\mathcal{A} \subseteq \mathcal{M}$ . Suppose  $(P_\alpha^{00})_{\alpha \in \mathcal{A}}$  is a weak  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$ . Then, for some monotonic  $\hat{\mathcal{A}} \leq \mathcal{A}$ ,  $\vec{\Gamma}$  has an  $(\hat{\mathcal{A}}, \delta, C'_B)$ -basis at  $(x_0, M_0, P^0)$ , with  $C'_B$  determined by  $C_B, C_w, m, n$ . Moreover, if  $\mathcal{A} \neq \emptyset$  and  $\max_{\alpha \in \mathcal{A}, \beta \in \mathcal{M}} \delta^{|\beta| - |\alpha|} |\partial^\beta P_\alpha^{00}(x_0)|$  exceeds a large enough constant determined by  $C_B, C_w, m, n$ , then we can take  $\hat{\mathcal{A}} < \mathcal{A}$  (strict inequality).*

**Proof.** If  $\mathcal{A}$  is empty, then we can take  $\hat{\mathcal{A}}$  empty. Thus, Lemma 7 holds trivially for  $\mathcal{A} = \emptyset$ . We suppose that  $\mathcal{A} \neq \emptyset$ .

Without loss of generality, we may take  $x_0 = 0$ . We introduce a constant  $\alpha > 0$  to be picked later, satisfying the

*Small  $\alpha$  condition:  $\alpha$  is less than a small enough constant determined by  $C_B, C_w, m, n$ .*

We write  $c, C, C'$ , etc., to denote constants determined by  $C_B, C_w, m, n$ ; and we write  $c(\alpha), C(\alpha), C'(\alpha)$ , etc., to denote constants determined by  $\alpha, C_B, C_w, m, n$ . These symbols may denote different constants in different occurrences.

Since  $(P_\alpha^{00})_{\alpha \in \mathcal{A}}$  is a weak  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}$  at  $(0, M_0, P^0)$ , we have the following.

$$(21) \quad P^0, P^0 \pm cM_0\delta^{m-|\alpha|}P_\alpha^{00} \in \Gamma(0, CM_0) \text{ for } \alpha \in \mathcal{A}.$$

$$(22) \quad \partial^\beta P_\alpha^{00}(0) = \delta_{\beta\alpha} \text{ for } \beta, \alpha \in \mathcal{A}.$$

$$(23) \quad |\partial^\beta P_\alpha^{00}(0)| \leq C\delta^{|\alpha| - |\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}, \beta \geq \alpha.$$

Thanks to (22), (23), the numbers  $F_{\alpha, \beta} = \delta^{|\beta| - |\alpha|} \partial^\beta P_\alpha^{00}(0)$  satisfy (13), (14), (15). Applying Lemma 6, we obtain real numbers  $\lambda_1, \dots, \lambda_n$  and a map  $\phi : \mathcal{A} \rightarrow \mathcal{M}$  satisfying (16),  $\dots$ , (19). We define a linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by setting

$$(24) \quad T(x_1, \dots, x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n) \text{ for } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

From (16)  $\dots$  (19) for our  $F_{\alpha, \beta}$ , we obtain the following.

$$(25) \quad c(\alpha) \leq \lambda_i \leq 1 \text{ for } i = 1, \dots, n.$$

$$(26) \quad \phi(\alpha) \leq \alpha \text{ for all } \alpha \in \mathcal{A}.$$

$$(27) \quad \text{For each } \alpha \in \mathcal{A}, \text{ either } \phi(\alpha) = \alpha \text{ or } \phi(\alpha) \notin \mathcal{A}.$$

$$(28) \quad \delta^{|\beta|-|\alpha|} |\partial^\beta (P_\alpha^{00} \circ T)(0)| \leq \mathfrak{a} \cdot \delta^{|\Phi(\alpha)|-|\alpha|} |\partial^{\Phi(\alpha)} (P_\alpha^{00} \circ T)(0)| \text{ for all } \alpha \in \mathcal{A}, \beta \in \mathcal{M} \setminus \Phi(\alpha).$$

Since the left-hand side of (28) is equal to  $\lambda^\beta \geq c(\mathfrak{a})$  for  $\beta = \alpha$  (by (22) and (25)), it follows from (28) that

$$(29) \quad \delta^{|\Phi(\alpha)|-|\alpha|} |\partial^{\Phi(\alpha)} (P_\alpha^{00} \circ T)(0)| \geq c(\mathfrak{a}) \text{ for } \alpha \in \mathcal{A}.$$

We define

$$(30) \quad \bar{\mathcal{A}} = \Phi(\mathcal{A})$$

and introduce a map

$$(31) \quad \psi : \bar{\mathcal{A}} \rightarrow \mathcal{A}$$

such that

$$(32) \quad \Phi(\psi(\bar{\alpha})) = \bar{\alpha} \text{ for all } \bar{\alpha} \in \bar{\mathcal{A}}.$$

Thanks to (26), (27), and Lemma 3.3 in [10] (mentioned above), we have

$$\bar{\mathcal{A}} \leq \mathcal{A},$$

with equality only when  $\Phi = \text{identity}$ . Moreover, suppose  $\Phi = \text{identity}$ . Then (22) and (28) show that

$$\begin{aligned} \left| \lambda^\beta \delta^{|\beta|-|\alpha|} \partial^\beta P_\alpha^{00}(0) \right| &= \delta^{|\beta|-|\alpha|} |\partial^\beta (P_\alpha^{00} \circ T)(0)| \\ &\leq |\partial^\alpha (P_\alpha^{00} \circ T)(0)| \\ &= \lambda^\alpha \end{aligned}$$

for  $\alpha \in \mathcal{A}$  and  $\beta \in \mathcal{M}$ . Hence, (25) yields

$$(33) \quad \delta^{|\beta|-|\alpha|} |\partial^\beta P_\alpha^{00}(0)| \leq C(\mathfrak{a}) \text{ for all } \alpha \in \mathcal{A} \text{ and } \beta \in \mathcal{M};$$

this holds provided  $\Phi = \text{identity}$ .

If  $\max_{\alpha \in \mathcal{A}, \beta \in \mathcal{M}} \delta^{|\beta|-|\alpha|} |\partial^\beta P_\alpha^{00}(0)| > C(\mathfrak{a})$  with  $C(\mathfrak{a})$  as in (33), then  $\Phi$  cannot be the identity map, and therefore  $\bar{\mathcal{A}} < \mathcal{A}$  (strict inequality).

Thus,

$$(34) \quad \bar{\mathcal{A}} \leq \mathcal{A}, \text{ with strict inequality if } \max_{\alpha \in \mathcal{A}, \beta \in \mathcal{M}} \delta^{|\beta|-|\alpha|} |\partial^\beta P_\alpha^{00}(0)| \text{ exceeds a large enough constant determined by } \mathfrak{a}, C_w, C_B, \mathfrak{m}, \mathfrak{n}.$$

For  $\bar{\alpha} \in \bar{\mathcal{A}}$ , we define

$$(35) \quad \mathfrak{b}_{\bar{\alpha}} = \left[ \delta^{|\bar{\alpha}|-|\psi(\bar{\alpha})|} \partial^{\bar{\alpha}} \left( P_{\psi(\bar{\alpha})}^{00} \circ T \right)(0) \right]^{-1};$$

estimate (29) with  $\alpha = \psi(\bar{\alpha})$  gives

$$(36) \quad |\mathbf{b}_{\bar{\alpha}}| \leq C(\mathbf{a}) \text{ for all } \bar{\alpha} \in \bar{\mathcal{A}}.$$

For  $\bar{\alpha} \in \bar{\mathcal{A}}$ , we also define

$$(37) \quad \bar{\mathbf{P}}_{\bar{\alpha}} = \mathbf{b}_{\bar{\alpha}} \delta^{|\bar{\alpha}| - |\psi(\bar{\alpha})|} \cdot \mathbf{P}_{\psi(\bar{\alpha})}^{00}.$$

From (28), (35), (37), we find that

$$(38) \quad \left| \delta^{|\beta| - |\bar{\alpha}|} \partial^\beta (\bar{\mathbf{P}}_{\bar{\alpha}} \circ \mathbf{T})(0) - \delta_{\beta \bar{\alpha}} \right| \leq \mathbf{a} \text{ for } \bar{\alpha} \in \bar{\mathcal{A}}, \beta \in \mathcal{M}.$$

Note that  $\delta^{m - |\bar{\alpha}|} \bar{\mathbf{P}}_{\bar{\alpha}} = \mathbf{b}_{\bar{\alpha}} \delta^{m - |\psi(\bar{\alpha})|} \mathbf{P}_{\psi(\bar{\alpha})}^{00}$ . Hence, (36) and (21) (with  $\alpha = \psi(\bar{\alpha})$ ) tell us that

$$(39) \quad \mathbf{P}^0 \pm c(\mathbf{a}) M_0 \delta^{m - |\bar{\alpha}|} \bar{\mathbf{P}}_{\bar{\alpha}} \in \Gamma(0, \mathbf{CM}_0),$$

since  $\Gamma(0, \mathbf{CM}_0)$  is convex.

Next, define

$$(40) \quad \hat{\mathcal{A}} = \{\gamma \in \mathcal{M} : \gamma = \bar{\alpha} + \bar{\gamma} \text{ for some } \bar{\alpha} \in \bar{\mathcal{A}}, \bar{\gamma} \in \mathcal{M}\},$$

and introduce maps  $\chi : \hat{\mathcal{A}} \rightarrow \bar{\mathcal{A}}$ ,  $\omega : \hat{\mathcal{A}} \rightarrow \mathcal{M}$ , such that

$$(41) \quad \hat{\alpha} = \chi(\hat{\alpha}) + \omega(\hat{\alpha}) \text{ for all } \hat{\alpha} \in \hat{\mathcal{A}}.$$

By definition (40),

$$(42) \quad \hat{\mathcal{A}} \text{ is monotonic,}$$

and  $\bar{\mathcal{A}} \subseteq \hat{\mathcal{A}}$ , hence

$$(43) \quad \hat{\mathcal{A}} \leq \bar{\mathcal{A}}.$$

From (34) and (43), we have

$$(44) \quad \hat{\mathcal{A}} \leq \mathcal{A}, \text{ with strict inequality if } \max_{\alpha \in \mathcal{A}, \beta \in \mathcal{M}} \delta^{|\beta| - |\alpha|} |\partial^\beta \mathbf{P}_\alpha^{00}(0)| \text{ exceeds a large enough constant determined by } \mathbf{a}, \mathbf{C}_w, \mathbf{C}_B, \mathbf{m}, \mathbf{n}.$$

For  $\hat{\alpha} \in \hat{\mathcal{A}}$ , we introduce the monomial

$$(45) \quad S_{\hat{\alpha}}(\mathbf{x}) = \frac{\chi(\hat{\alpha})!}{\hat{\alpha}!} \lambda^{-\omega(\hat{\alpha})} \mathbf{x}^{\omega(\hat{\alpha})} \quad (\mathbf{x} \in \mathbb{R}^n).$$

We have

$$(46) \quad S_{\hat{\alpha}} \circ \mathbf{T}(\mathbf{x}) = \frac{\chi(\hat{\alpha})!}{\hat{\alpha}!} \mathbf{x}^{\omega(\hat{\alpha})},$$

hence

$$(47) \quad \partial^\beta (S_{\hat{\alpha}} \circ \mathbf{T})(0) = \frac{\chi(\hat{\alpha})! \omega(\hat{\alpha})!}{\hat{\alpha}!} \delta_{\beta \omega(\hat{\alpha})} \text{ for } \hat{\alpha} \in \hat{\mathcal{A}}, \beta \in \mathcal{M}.$$



We study the derivatives  $\partial^\beta \left( [(S_{\hat{\alpha}} \bar{P}_{\chi(\hat{\alpha})}) \circ T] (0) \right)$  for  $\hat{\alpha} \in \hat{\mathcal{A}}, \beta \in \mathcal{M}$ .

Case 1: If  $\beta$  is not of the form  $\beta = \omega(\hat{\alpha}) + \tilde{\beta}$  for some  $\tilde{\beta} \in \mathcal{M}$ , then (47) gives

$$(48) \quad \partial^\beta \left[ (S_{\hat{\alpha}} \bar{P}_{\chi(\hat{\alpha})}) \circ T \right] (0) = 0.$$

Case 2: Suppose  $\beta = \omega(\hat{\alpha}) + \tilde{\beta}$  for some  $\tilde{\beta} \in \mathcal{M}$ . Then (47) gives

$$\begin{aligned} & \partial^\beta \left[ (S_{\hat{\alpha}} \bar{P}_{\chi(\hat{\alpha})}) \circ T \right] (0) \\ &= \frac{\beta!}{\omega(\hat{\alpha})! \tilde{\beta}!} \left[ \partial^{\omega(\hat{\alpha})} (S_{\hat{\alpha}} \circ T) (0) \right] \cdot \left[ \partial^{\tilde{\beta}} (\bar{P}_{\chi(\hat{\alpha})} \circ T) (0) \right] \\ &= \frac{\beta!}{\omega(\hat{\alpha})! \tilde{\beta}!} \frac{\chi(\hat{\alpha})! \omega(\hat{\alpha})!}{\hat{\alpha}!} \cdot \left[ \partial^{\tilde{\beta}} (\bar{P}_{\chi(\hat{\alpha})} \circ T) (0) \right] \\ &= \frac{\beta! \chi(\hat{\alpha})!}{\hat{\alpha}! \tilde{\beta}!} \left[ \partial^{\tilde{\beta}} (\bar{P}_{\chi(\hat{\alpha})} \circ T) (0) \right]. \end{aligned}$$

Hence, by (38), we have

$$(49) \quad \left| \frac{\hat{\alpha}! \tilde{\beta}!}{\beta! \chi(\hat{\alpha})!} \delta^{|\tilde{\beta}| - |\chi(\hat{\alpha})|} \partial^\beta \left[ (S_{\hat{\alpha}} \cdot \bar{P}_{\chi(\hat{\alpha})}) \circ T \right] (0) - \delta_{\tilde{\beta} \chi(\hat{\alpha})} \right| \leq \mathfrak{a}.$$

Since in this case  $\beta = \omega(\hat{\alpha}) + \tilde{\beta}$  and  $\hat{\alpha} = \omega(\hat{\alpha}) + \chi(\hat{\alpha})$  (see (41)), we have  $\delta_{\tilde{\beta} \chi(\hat{\alpha})} = \delta_{\beta \hat{\alpha}}$ ,  $|\tilde{\beta}| - |\chi(\hat{\alpha})| = |\beta| - |\hat{\alpha}|$ , and  $\frac{\hat{\alpha}! \tilde{\beta}!}{\beta! \chi(\hat{\alpha})!} = 1$  if  $\beta = \hat{\alpha}$ .

Hence, (49) implies that

$$(50) \quad \left| \delta^{|\beta| - |\hat{\alpha}|} \partial^\beta \left[ (S_{\hat{\alpha}} \cdot \bar{P}_{\chi(\hat{\alpha})}) \circ T \right] (0) - \delta_{\beta \hat{\alpha}} \right| \leq C \mathfrak{a}$$

in Case 2.

Thanks to (41) and (48), estimate (50) holds also in Case 1.

Thus, (50) holds for all  $\hat{\alpha} \in \hat{\mathcal{A}}, \beta \in \mathcal{M}$ . Consequently,

$$(51) \quad \left| \delta^{|\beta| - |\hat{\alpha}|} \partial^\beta \left[ [S_{\hat{\alpha}} \odot_0 \bar{P}_{\chi(\hat{\alpha})}] \circ T \right] (0) - \delta_{\beta \hat{\alpha}} \right| \leq C \mathfrak{a}$$

for all  $\hat{\alpha} \in \hat{\mathcal{A}}, \beta \in \mathcal{M}$ .

We prepare to apply Lemma 5, with  $\hat{S} = \delta^{-|\omega(\hat{\alpha})|} S_{\hat{\alpha}}$  and  $\hat{P} = M_0 \delta^{m - |\chi(\hat{\alpha})|} \bar{P}_{\chi(\hat{\alpha})}$ . From (25) and (45), we have

$$(52) \quad \left| \partial^\beta \left( \delta^{-|\omega(\hat{\alpha})|} S_{\hat{\alpha}} \right) (0) \right| \leq C(\mathfrak{a}) \delta^{-|\beta|} \text{ for } \hat{\alpha} \in \hat{\mathcal{A}}, \beta \in \mathcal{M}.$$

From (38) we have

$$\left| \partial^\beta (\bar{P}_{\chi(\hat{\alpha})} \circ T) (0) \right| \leq C \delta^{|\chi(\hat{\alpha})| - |\beta|} \text{ for } \hat{\alpha} \in \hat{\mathcal{A}}, \beta \in \mathcal{M};$$

hence, by (25),

$$(53) \quad \left| \partial^\beta \left( M_0 \delta^{m - |\chi(\hat{\alpha})|} \bar{P}_{\chi(\hat{\alpha})} \right) (0) \right| \leq C(\mathfrak{a}) M_0 \delta^{m - |\beta|} \text{ for } \hat{\alpha} \in \hat{\mathcal{A}}, \beta \in \mathcal{M}.$$

Also, (39) gives

$$(54) \quad P^0 \pm c(a) M_0 \delta^{m-|x(\hat{\alpha})|} \bar{P}_{X(\hat{\alpha})} \in \Gamma(0, CM_0) \text{ for } \hat{\alpha} \in \hat{\mathcal{A}}.$$

Our results (52), (53), (54) are the hypotheses of Lemma 5 for  $\hat{S}, \hat{P}$  as given above. Applying that lemma, we learn that

$$P^0 \pm c(a) \left( \delta^{-|\omega(\hat{\alpha})|} S_{\hat{\alpha}} \right) \odot_0 \left( M_0 \delta^{m-|x(\hat{\alpha})|} \bar{P}_{X(\hat{\alpha})} \right) \in \Gamma(0, C(a) M_0) \text{ for } \hat{\alpha} \in \hat{\mathcal{A}}.$$

Recalling (41), we conclude that

$$(55) \quad P^0 \pm c(a) M_0 \delta^{m-|\hat{\alpha}|} S_{\hat{\alpha}} \odot_0 \bar{P}_{X(\hat{\alpha})} \in \Gamma(0, C(a) M_0) \text{ for } \hat{\alpha} \in \hat{\mathcal{A}}.$$

Next, (51) and the *small a condition* tell us that there exists a matrix of real numbers  $(b_{\gamma\hat{\alpha}})_{\gamma, \hat{\alpha} \in \hat{\mathcal{A}}}$  satisfying

$$(56) \quad \sum_{\hat{\alpha} \in \hat{\mathcal{A}}} b_{\gamma\hat{\alpha}} \partial^\beta \left( [S_{\hat{\alpha}} \odot_0 \bar{P}_{X(\hat{\alpha})}] \circ T \right) (0) \cdot \delta^{|\beta|-|\hat{\alpha}|} = \delta_{\gamma\beta} \text{ for } \gamma, \beta \in \hat{\mathcal{A}}$$

and

$$(57) \quad |b_{\gamma\hat{\alpha}}| \leq 2, \text{ for all } \gamma, \hat{\alpha} \in \hat{\mathcal{A}}.$$

From (56) we have

$$(58) \quad \partial^\beta \left\{ \sum_{\hat{\alpha} \in \hat{\mathcal{A}}} b_{\gamma\hat{\alpha}} \delta^{|\gamma|-|\hat{\alpha}|} \left( [S_{\hat{\alpha}} \odot_0 \bar{P}_{X(\hat{\alpha})}] \circ T \right) \right\} (0) = \delta_{\gamma\beta} \text{ for } \gamma, \beta \in \hat{\mathcal{A}}.$$

Also, (51) and (57) imply that

$$(59) \quad \left| \partial^\beta \left\{ \sum_{\hat{\alpha} \in \hat{\mathcal{A}}} b_{\gamma\hat{\alpha}} \delta^{|\gamma|-|\hat{\alpha}|} \left( [S_{\hat{\alpha}} \odot_0 \bar{P}_{X(\hat{\alpha})}] \circ T \right) \right\} (0) \right| \leq C \delta^{|\gamma|-|\beta|} \text{ for } \gamma \in \hat{\mathcal{A}}, \beta \in \mathcal{M}.$$

Since

$$\begin{aligned} & M_0 \delta^{m-|\gamma|} \left\{ \sum_{\hat{\alpha} \in \hat{\mathcal{A}}} b_{\gamma\hat{\alpha}} \delta^{|\gamma|-|\hat{\alpha}|} \left( [S_{\hat{\alpha}} \odot_0 \bar{P}_{X(\hat{\alpha})}] \right) \right\} \\ &= \sum_{\hat{\alpha} \in \hat{\mathcal{A}}} b_{\gamma\hat{\alpha}} \cdot \left\{ M_0 \delta^{m-|\hat{\alpha}|} \cdot [S_{\hat{\alpha}} \odot_0 \bar{P}_{X(\hat{\alpha})}] \right\}, \end{aligned}$$

we learn from (55), (57) and the Trivial Remark on Convex Sets in Section I.1, that

$$(60) \quad P^0 \pm c(a) M_0 \delta^{m-|\gamma|} \left\{ \sum_{\hat{\alpha} \in \hat{\mathcal{A}}} b_{\gamma\hat{\alpha}} \delta^{|\gamma|-|\hat{\alpha}|} \left( [S_{\hat{\alpha}} \odot_0 \bar{P}_{X(\hat{\alpha})}] \right) \right\} \in \Gamma(0, C(a) M_0)$$

for  $\gamma \in \hat{\mathcal{A}}$ .

We define

$$(61) \quad P_\gamma = \lambda^\gamma \sum_{\hat{\alpha} \in \hat{\mathcal{A}}} b_{\gamma \hat{\alpha}} \delta^{|\gamma| - |\hat{\alpha}|} [S_{\hat{\alpha}} \odot_0 \bar{P}_{X(\hat{\alpha})}] \text{ for } \gamma \in \hat{\mathcal{A}}.$$

From (25), we have  $|\lambda^\gamma| \leq C(\alpha)$ . Recall from (21) that  $P^0 \in \Gamma(0, CM_0)$ . Hence, we deduce from (60) (and from the convexity of  $\Gamma(0, C(\alpha)M_0)$ ) that

$$(62) \quad P^0, P^0 + c(\alpha)M_0\delta^{m-|\gamma|}P_\gamma, P^0 - c(\alpha)M_0\delta^{m-|\gamma|}P_\gamma \in \Gamma(0, C(\alpha)M_0) \text{ for } \gamma \in \hat{\mathcal{A}}.$$

Also, (58) and (61) give

$$(63) \quad \partial^\beta P_\gamma(0) = \delta_{\beta\gamma} \text{ for } \beta, \gamma \in \hat{\mathcal{A}}.$$

From (25), (59), (61), we have

$$(64) \quad |\partial^\beta P_\gamma(0)| \leq C(\alpha) \delta^{|\gamma| - |\beta|} \text{ for } \gamma \in \hat{\mathcal{A}}, \beta \in \mathcal{M}.$$

Our results (62), (63), (64) show that

$$(65) \quad (P_\gamma)_{\gamma \in \hat{\mathcal{A}}} \text{ is an } (\hat{\mathcal{A}}, \delta, C(\alpha))\text{-basis for } \vec{\Gamma} \text{ at } (0, M_0, P^0).$$

We now pick  $\alpha$  to be a constant determined by  $C_B, C_w, m, n$ , small enough to satisfy our *small  $\alpha$  condition*. Then (42), (44), and (65) immediately imply the conclusions of Lemma 7.

The proof of that lemma is complete. ■

The next result is a consequence of the Relabeling Lemma (Lemma 7).

**Lemma 8 (Control  $\Gamma$  Using Basis)** *Let  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field. Let  $x_0 \in E$ ,  $M_0 > 0$ ,  $0 < \delta \leq \delta_{\max}$ ,  $C_B > 0$ ,  $\mathcal{A} \subseteq \mathcal{M}$ , and let  $P, P^0 \in \mathcal{P}$ . Suppose  $\vec{\Gamma}$  has an  $(\mathcal{A}, \delta, C_B)$ -basis at  $(x_0, M_0, P^0)$ . Suppose also that*

$$(66) \quad P \in \Gamma(x_0, C_B M_0),$$

$$(67) \quad \partial^\beta (P - P^0)(x_0) = 0 \text{ for all } \beta \in \mathcal{A}, \text{ and}$$

$$(68) \quad \max_{\beta \in \mathcal{M}} \delta^{|\beta|} |\partial^\beta (P - P^0)(x_0)| \geq M_0 \delta^m.$$

*Then there exist  $\hat{\mathcal{A}} \subseteq \mathcal{M}$  and  $\hat{P}^0 \in \mathcal{P}$  with the following properties.*

$$(69) \quad \hat{\mathcal{A}} \text{ is monotonic.}$$

$$(70) \quad \hat{\mathcal{A}} < \mathcal{A} \text{ (strict inequality).}$$

$$(71) \quad \vec{\Gamma} \text{ has an } (\hat{\mathcal{A}}, \delta, C'_B)\text{-basis at } (x_0, M_0, \hat{P}^0), \text{ with } C'_B \text{ determined by } C_B, C_w, m, n.$$

$$(72) \quad \partial^\beta (\hat{P}^0 - P^0)(x_0) = 0 \text{ for all } \beta \in \mathcal{A}.$$

$$(73) \quad |\partial^\beta (\hat{P}^0 - P^0)(x_0)| \leq M_0 \delta^{m-|\beta|} \text{ for all } \beta \in \mathcal{M}.$$

**Proof.** We write  $c, C, C'$ , etc., to denote constants determined by  $C_B, C_w, m, n$ . These symbols may denote different constants in different occurrences.

Let  $(P_\alpha)_{\alpha \in \mathcal{A}}$  be an  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, P^0)$ . By definition,

$$(74) \quad P^0 \in \Gamma(x_0, C_B M_0),$$

$$(75) \quad P^0 \pm c M_0 \delta^{m-|\alpha|} P_\alpha \in \Gamma(x_0, C_B M) \text{ for all } \alpha \in \mathcal{A},$$

$$(76) \quad \partial^\beta P_\alpha(x_0) = \delta_{\beta\alpha} \text{ for } \beta, \alpha \in \mathcal{A},$$

$$(77) \quad |\partial^\beta P_\alpha(x_0)| \leq C \delta^{|\alpha|-|\beta|} \text{ for all } \alpha \in \mathcal{A}, \beta \in \mathcal{M}.$$

Assuming there exists  $P \in \mathcal{P}$  such that (66), (67), (68) hold, then there exists  $P$  such that (66), (67) hold, and also

$$(78) \quad \max_{\beta \in \mathcal{M}} \delta^{|\beta|} |\partial^\beta (P - P^0)(x_0)| = M_0 \delta^m.$$

(We just take our new  $P$  to be a convex combination of our original  $P$  and  $P_0$ .)

We pick  $\gamma \in \mathcal{M}$  to achieve the above max. Thanks to (67), we have

$$(79) \quad \gamma \notin \mathcal{A},$$

hence

$$(80) \quad \mathcal{A} \cup \{\gamma\} < \mathcal{A} \text{ (strict inequality).}$$

We set

$$(81) \quad \hat{P}^0 = \frac{1}{2}(P^0 + P).$$

From (66), (74), (75), we have

$$(82) \quad \hat{P}^0, \hat{P}^0 \pm c' M_0 \delta^{m-|\alpha|} P_\alpha \in \Gamma(x_0, C M_0) \text{ for } \alpha \in \mathcal{A}.$$

Also,

$$(83) \quad \hat{P}^0 \pm \frac{1}{2}(P - P^0) \in \Gamma(x_0, C M)$$

since  $\hat{P}^0 + \frac{1}{2}(P - P^0) = P$  and  $\hat{P}^0 - \frac{1}{2}(P - P^0) = P^0$ .

From (67) and (78), we have

$$(84) \quad \partial^\beta (\hat{P}^0 - P^0)(x_0) = 0 \text{ for all } \beta \in \mathcal{A},$$

and

$$(85) \quad |\partial^\beta (\hat{P}^0 - P^0)(x_0)| \leq M_0 \delta^{m-|\beta|} \text{ for all } \beta \in \mathcal{M}.$$

We define

$$(86) \quad P_\gamma^\# = [\partial^\gamma (P - P^0)(x_0)]^{-1} \cdot (P - P^0).$$

We are not dividing by zero here; by (78) and the definition of  $\gamma$ , we have

$$(87) \quad |\partial^\gamma (P - P^0)(x_0)|^{-1} = M_0^{-1} \delta^{|\gamma|-m}.$$

From (67), (79), (86), we have

$$(88) \quad \partial^\beta P_\gamma^\#(x_0) = \delta_{\beta\gamma} \text{ for all } \beta \in \mathcal{A} \cup \{\gamma\}.$$

Also, (78), (86), (87) give

$$(89) \quad |\partial^\beta P_\gamma^\#(x_0)| \leq M_0^{-1} \delta^{|\gamma|-m} \cdot M_0 \delta^{m-|\beta|} = \delta^{|\gamma|-|\beta|} \text{ for all } \beta \in \mathcal{M}.$$

From (86), (87), we have  $P - P^0 = \sigma M_0 \delta^{m-|\gamma|} P_\gamma^\#$  for  $\sigma = 1$  or  $\sigma = -1$ . Therefore, (83) implies that

$$(90) \quad \hat{P}^0 \pm c M_0 \delta^{m-|\gamma|} P_\gamma^\# \in \Gamma(x_0, CM_0).$$

From (82), (90) and the Trivial Remark on Convex Sets in Section I.1, we conclude that

$$(91) \quad \hat{P}^0 + s M_0 \delta^{m-|\gamma|} P_\gamma^\# + \sum_{\alpha \in \mathcal{A}} t_\alpha \cdot M_0 \delta^{m-|\alpha|} P_\alpha \in \Gamma(x_0, CM_0),$$

whenever  $|s|, |t_\alpha| \leq c$  (all  $\alpha \in \mathcal{A}$ ) for a small enough  $c$ .

For  $\alpha \in \mathcal{A}$ , we define

$$(92) \quad P_\alpha^\# = P_\alpha - [\partial^\gamma P_\alpha(x_0)] \cdot P_\gamma^\#.$$

Fix  $\alpha \in \mathcal{A}$ . If  $\beta \in \mathcal{A}$ , then (76), (79), (88) imply

$$\partial^\beta P_\alpha^\#(x_0) = \partial^\beta P_\alpha(x_0) - [\partial^\gamma P_\alpha(x_0)] \cdot \partial^\beta P_\gamma^\#(x_0) = \delta_{\beta\alpha}.$$

On the other hand, (79) and (88) yield

$$\partial^\gamma P_\alpha^\#(x_0) = \partial^\gamma P_\alpha(x_0) - [\partial^\gamma P_\alpha(x_0)] \cdot \partial^\gamma P_\gamma^\#(x_0) = 0 = \delta_{\gamma\alpha}.$$

Thus,

$$\partial^\beta P_\alpha^\#(x_0) = \delta_{\beta\alpha} \text{ for } \beta \in \mathcal{A} \cup \{\gamma\}, \alpha \in \mathcal{A}.$$

Together with (88), this tells us that

$$(93) \quad \partial^\beta P_\alpha^\#(x_0) = \delta_{\beta\alpha} \text{ for } \beta, \alpha \in \mathcal{A} \cup \{\gamma\}.$$

Next, we learn from (77), (89), (92) that

$$\begin{aligned} |\partial^\beta P_\alpha^\#(x_0)| &\leq |\partial^\beta P_\alpha(x_0)| + |\partial^\gamma P_\alpha(x_0)| \cdot |\partial^\beta P_\gamma^\#(x_0)| \\ &\leq C\delta^{|\alpha|-|\beta|} + C\delta^{|\alpha|-|\gamma|} \cdot \delta^{|\gamma|-|\beta|} \\ &\leq C'\delta^{|\alpha|-|\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}. \end{aligned}$$

Together with (89), this tells us that

$$(94) \quad \left| \partial^\beta P_\alpha^\#(x_0) \right| \leq C\delta^{|\alpha|-|\beta|} \text{ for all } \alpha \in \mathcal{A} \cup \{\gamma\}, \beta \in \mathcal{M}.$$

Next, note that for  $\alpha \in \mathcal{A}$ , we have

$$M_0\delta^{m-|\alpha|}P_\alpha^\# = M_0\delta^{m-|\alpha|}P_\alpha - \left[ \delta^{|\gamma|-|\alpha|}\partial^\gamma P_\alpha(x_0) \right] \cdot M_0\delta^{m-|\gamma|}P_\gamma^\#,$$

with  $\left| \left[ \delta^{|\gamma|-|\alpha|}\partial^\gamma P_\alpha(x_0) \right] \right| \leq C$  by (77).

Therefore, (91) shows that

$$\hat{P}^0 \pm cM_0\delta^{m-|\alpha|}P_\alpha^\# \in \Gamma(x_0, CM_0) \text{ for } \alpha \in \mathcal{A},$$

provided we take  $c$  small enough. Together with (90), this yields

$$(95) \quad \hat{P}^0 \pm cM_0\delta^{m-|\alpha|}P_\alpha^\# \in \Gamma(x_0, CM_0) \text{ for all } \alpha \in \mathcal{A} \cup \{\gamma\}.$$

Our results (93), (94), (95) tell us that  $\left( P_\alpha^\# \right)_{\alpha \in \mathcal{A} \cup \{\gamma\}}$  is an  $(\mathcal{A} \cup \{\gamma\}, \delta, C)$ -basis for  $\vec{\Gamma}$  at  $(x_0, M_0, \hat{P}^0)$ .

Consequently, the Relabeling Lemma (Lemma 7) produces a set  $\hat{\mathcal{A}} \subseteq \mathcal{M}$  with the following properties.

$$(96) \quad \hat{\mathcal{A}} \text{ is monotonic.}$$

$$(97) \quad \hat{\mathcal{A}} \leq \mathcal{A} \cup \{\gamma\} < \mathcal{A}, \text{ see (80).}$$

$$(98) \quad \vec{\Gamma} \text{ has an } (\hat{\mathcal{A}}, \delta, C')\text{-basis at } (x_0, M_0, \hat{P}^0).$$

Our results (84), (85), (96), (97), (98) are the conclusions (69)···(73) of Lemma 8.

The proof of that lemma is complete. ■

## I.4 The Transport Lemma

In this section, we prove the following result.

**Lemma 9 (Transport Lemma)** *Let  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  be a shape field. For  $l \geq 1$ , let  $\vec{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0}$  be the  $l$ -th refinement of  $\vec{\Gamma}_0$ .*

(1) Suppose  $\mathcal{A} \subseteq \mathcal{M}$  is monotonic and  $\hat{\mathcal{A}} \subseteq \mathcal{M}$  (not necessarily monotonic).

Let  $x_0 \in E$ ,  $M_0 > 0$ ,  $l_0 \geq 1$ ,  $\delta > 0$ ,  $C_B$ ,  $\hat{C}_B$ ,  $C_{\text{DIFF}} > 0$ . Let  $P^0, \hat{P}^0 \in \mathcal{P}$ . Assume that the following hold.

(2)  $\vec{\Gamma}_{l_0}$  has an  $(\mathcal{A}, \delta, C_B)$ -basis at  $(x_0, M_0, P^0)$ , and an  $(\hat{\mathcal{A}}, \delta, \hat{C}_B)$ -basis at  $(x_0, M_0, \hat{P}^0)$ .

(3)  $\partial^\beta (P^0 - \hat{P}^0) \equiv 0$  for  $\beta \in \mathcal{A}$ .

(4)  $|\partial^\beta (P^0 - \hat{P}^0)(x_0)| \leq C_{\text{DIFF}} M_0 \delta^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ .

Let  $y_0 \in E$ , and suppose that

(5)  $|x_0 - y_0| \leq \epsilon_0 \delta$ ,

where  $\epsilon_0$  is a small enough constant determined by  $C_B$ ,  $\hat{C}_B$ ,  $C_{\text{DIFF}}$ ,  $m$ ,  $n$ . Then there exists  $\hat{P}^\# \in \mathcal{P}$  with the following properties.

(6)  $\vec{\Gamma}_{l_0-1}$  has both an  $(\mathcal{A}, \delta, C'_B)$ -basis and an  $(\hat{\mathcal{A}}, \delta, C'_B)$ -basis at  $(y_0, M_0, \hat{P}^\#)$ , with  $C'_B$  determined by  $C_B$ ,  $\hat{C}_B$ ,  $C_{\text{DIFF}}$ ,  $m$ ,  $n$ .

(7)  $\partial^\beta (\hat{P}^\# - P^0) \equiv 0$  for  $\beta \in \mathcal{A}$ .

(8)  $|\partial^\beta (\hat{P}^\# - P^0)(x_0)| \leq C' M_0 \delta^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ , with  $C'$  determined by  $C_B$ ,  $\hat{C}_B$ ,  $C_{\text{DIFF}}$ ,  $m$ ,  $n$ .

**Remark** Note that  $\mathcal{A}$  and  $\hat{\mathcal{A}}$  play different rôles here; see (1), (3), and (7).

**Proof of the Transport Lemma.** In the trivial case  $\mathcal{A} = \hat{\mathcal{A}} = \emptyset$ , the Transport Lemma holds simply because (by definition of the  $l$ -th refinement) there exists  $\hat{P}^\# \in \Gamma_{l_0-1}(y_0, C_B M_0)$  such that

$$\begin{aligned} |\partial^\beta (\hat{P}^\# - P^0)(x_0)| &\leq C_B M_0 |x_0 - y_0|^{m-|\beta|} \\ &\leq C_B M_0 \delta^{m-|\beta|} \text{ for } \beta \in \mathcal{M}. \end{aligned}$$

(Recall that  $P^0 \in \Gamma_{l_0}(x_0, C_B M_0)$  since  $\vec{\Gamma}_{l_0}$  has an  $(\mathcal{A}, \delta, C_B)$ -basis at  $(x_0, M_0, P^0)$ .)

From now on, we suppose that

(9)  $\#(\mathcal{A}) + \#(\hat{\mathcal{A}}) \neq 0$ .

In proving the Transport Lemma, we do not yet take  $\epsilon_0$  to be a constant determined by  $C_B$ ,  $\hat{C}_B$ ,  $C_{\text{DIFF}}$ ,  $m$ ,  $n$ . Rather, we make the following

(10) *Small  $\epsilon_0$  assumption:*  $\epsilon_0$  is less than a small enough constant determined by  $C_B$ ,  $\hat{C}_B$ ,  $C_{\text{DIFF}}$ ,  $m$ ,  $n$ .

Assuming (1)⋯(5) and (9), (10), we will prove that there exists  $\hat{P}^\# \in \mathcal{P}$  satisfying (6), (7), (8). Once we do so, we may then pick  $\epsilon_0$  to be a constant determined by  $C_B, \hat{C}_B, C_{\text{DIFF}}, m, n$ , small enough to satisfy (10). That will complete the proof of the Transport Lemma.

Thus, assume (1)⋯(5) and (9), (10).

We write  $c, C, C'$ , etc. to denote “controlled constants”, i.e., constants determined by  $C_B, \hat{C}_B, C_{\text{DIFF}}, m, n$ . These symbols may denote different controlled constants in different occurrences.

Let  $(P_\alpha)_{\alpha \in \mathcal{A}}$  be an  $(\mathcal{A}, \delta, C_B)$ -basis for  $\vec{\Gamma}_{l_0}$  at  $(x_0, M_0, P^0)$ , and let  $(\hat{P}_\alpha)_{\alpha \in \hat{\mathcal{A}}}$  be an  $(\hat{\mathcal{A}}, \delta, \hat{C}_B)$ -basis for  $\vec{\Gamma}_{l_0}$  at  $(x_0, M_0, \hat{P}^0)$ .

By definition, the following hold.

$$(11) \quad P^0 + c_0 \sigma M_0 \delta^{m-|\alpha|} P_\alpha \in \Gamma_{l_0}(x_0, CM_0) \text{ for } \alpha \in \mathcal{A}, \sigma \in \{1, -1\}.$$

$$(12) \quad \hat{P}^0 + \hat{c}_0 \sigma M_0 \delta^{m-|\alpha|} \hat{P}_\alpha \in \Gamma_{l_0}(x_0, CM_0) \text{ for } \alpha \in \hat{\mathcal{A}}, \sigma \in \{1, -1\}.$$

$$(13) \quad \partial^\beta P_\alpha(x_0) = \delta_{\beta\alpha} \text{ for } \alpha, \beta \in \mathcal{A}.$$

$$(14) \quad \partial^\beta \hat{P}_\alpha(x_0) = \delta_{\beta\alpha} \text{ for } \alpha, \beta \in \hat{\mathcal{A}}.$$

$$(15) \quad |\partial^\beta P_\alpha(x_0)| \leq C \delta^{|\alpha|-|\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}.$$

$$(16) \quad |\partial^\beta \hat{P}_\alpha(x_0)| \leq C \delta^{|\alpha|-|\beta|} \text{ for } \alpha \in \hat{\mathcal{A}}, \beta \in \mathcal{M}.$$

We fix controlled constants  $c_0, \hat{c}_0$  as in (11), (12). Recall that  $\vec{\Gamma}_{l_0}$  is the first refinement of  $\vec{\Gamma}_{l_0-1}$ . Therefore, by (5), (10), (11), there exists  $\tilde{P}_{\alpha,\sigma} \in \Gamma_{l_0-1}(y_0, CM_0)$  ( $\alpha \in \mathcal{A}, \sigma \in \{1, -1\}$ ) such that

$$\begin{aligned} & \left| \partial^\beta \left( \tilde{P}_{\alpha,\sigma} - \left[ P^0 + c_0 \sigma M_0 \delta^{m-|\alpha|} P_\alpha \right] \right) (x_0) \right| \\ & \leq CM_0 |x_0 - y_0|^{m-|\beta|} \leq C \epsilon_0 M_0 \delta^{m-|\beta|}, \text{ for } \beta \in \mathcal{M}. \end{aligned}$$

Writing

$$E_{\alpha,\sigma} = \frac{\tilde{P}_{\alpha,\sigma} - \left[ P^0 + c_0 \sigma M_0 \delta^{m-|\alpha|} P_\alpha \right]}{c_0 \sigma M_0 \delta^{m-|\alpha|}},$$

we have

$$(17) \quad P^0 + c_0 \sigma M_0 \delta^{m-|\alpha|} (P_\alpha + E_{\alpha,\sigma}) \in \Gamma_{l_0-1}(y_0, CM_0) \text{ for } \alpha \in \mathcal{A}, \sigma \in \{1, -1\},$$

and

$$(18) \quad |\partial^\beta E_{\alpha,\sigma}(x_0)| \leq C \epsilon_0 \delta^{|\alpha|-|\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}, \sigma \in \{1, -1\}.$$

Similarly, we obtain  $\hat{E}_{\alpha,\sigma} \in \mathcal{P}$  ( $\alpha \in \hat{\mathcal{A}}, \sigma \in \{1, -1\}$ ), satisfying

$$(19) \quad \hat{P}^0 + \hat{c}_0 \sigma M_0 \delta^{m-|\alpha|} (\hat{P}_\alpha + \hat{E}_{\alpha,\sigma}) \in \Gamma_{l_0-1}(y_0, CM_0) \text{ for } \alpha \in \hat{\mathcal{A}}, \sigma \in \{1, -1\},$$



and

$$(20) \quad |\partial^\beta \hat{\mathbf{E}}_{\alpha, \sigma}(x_0)| \leq C \epsilon_0 \delta^{|\alpha| - |\beta|} \text{ for } \alpha \in \hat{\mathcal{A}}, \beta \in \mathcal{M}, \sigma \in \{1, -1\}.$$

We introduce the following polynomials:

(21)

$$\hat{\mathbf{P}}' = \frac{1}{2} [\#(\mathcal{A}) + \#(\hat{\mathcal{A}})]^{-1} \left( \begin{array}{l} \sum_{\alpha \in \mathcal{A}, \sigma = \pm 1} \left\{ \mathbf{P}^0 + \mathbf{c}_0 \sigma M_0 \delta^{m-|\alpha|} (\mathbf{P}_\alpha + \mathbf{E}_{\alpha, \sigma}) \right\} \\ + \sum_{\alpha \in \hat{\mathcal{A}}, \sigma = \pm 1} \left\{ \hat{\mathbf{P}}^0 + \hat{\mathbf{c}}_0 \sigma M_0 \delta^{m-|\alpha|} (\hat{\mathbf{P}}_\alpha + \hat{\mathbf{E}}_{\alpha, \sigma}) \right\} \end{array} \right)$$

(see (9));

(22)

$$\begin{aligned} \mathbf{P}'_\alpha &= \frac{1}{2c_0 M_0 \delta^{m-|\alpha|}} \left( \begin{array}{l} \left\{ \mathbf{P}^0 + c_0 M_0 \delta^{m-|\alpha|} (\mathbf{P}_\alpha + \mathbf{E}_{\alpha, 1}) \right\} \\ - \left\{ \mathbf{P}^0 - c_0 M_0 \delta^{m-|\alpha|} (\mathbf{P}_\alpha + \mathbf{E}_{\alpha, -1}) \right\} \end{array} \right) \\ &= \mathbf{P}_\alpha + \frac{1}{2} (\mathbf{E}_{\alpha, 1} + \mathbf{E}_{\alpha, -1}) \text{ for } \alpha \in \mathcal{A}; \end{aligned}$$

(23)

$$\begin{aligned} \hat{\mathbf{P}}'_\alpha &= \frac{1}{2\hat{c}_0 M_0 \delta^{m-|\alpha|}} \left( \begin{array}{l} \left\{ \hat{\mathbf{P}}^0 + \hat{c}_0 M_0 \delta^{m-|\alpha|} (\hat{\mathbf{P}}_\alpha + \hat{\mathbf{E}}_{\alpha, 1}) \right\} \\ - \left\{ \hat{\mathbf{P}}^0 - \hat{c}_0 M_0 \delta^{m-|\alpha|} (\hat{\mathbf{P}}_\alpha + \hat{\mathbf{E}}_{\alpha, -1}) \right\} \end{array} \right) \\ &= \hat{\mathbf{P}}_\alpha + \frac{1}{2} (\hat{\mathbf{E}}_{\alpha, 1} + \hat{\mathbf{E}}_{\alpha, -1}) \text{ for } \alpha \in \hat{\mathcal{A}}. \end{aligned}$$

For a small enough controlled constant  $c_1$ , we have

$$(24) \quad \hat{\mathbf{P}}' + c_1 M_0 \delta^{m-|\alpha|} \mathbf{P}'_\alpha, \hat{\mathbf{P}}' - c_1 M_0 \delta^{m-|\alpha|} \mathbf{P}'_\alpha \in \Gamma_{l_0-1}(y_0, CM_0) \text{ for } \alpha \in \mathcal{A},$$

and

$$(25) \quad \hat{\mathbf{P}}' + c_1 M_0 \delta^{m-|\alpha|} \hat{\mathbf{P}}'_\alpha, \hat{\mathbf{P}}' - c_1 M_0 \delta^{m-|\alpha|} \hat{\mathbf{P}}'_\alpha \in \Gamma_{l_0-1}(y_0, CM_0) \text{ for } \alpha \in \hat{\mathcal{A}},$$

because each of the polynomials in (24), (25) is a convex combination of the polynomials in (17), (19). (In fact, we can take  $c_1 = \frac{c_0}{\#(\mathcal{A}) + \#(\hat{\mathcal{A}})}$ .)

From (24), (25) and the Trivial Remark on Convex Sets in Section I.1, we obtain the following, for a small enough controlled constant  $c_2$ .

$$(26) \quad \hat{\mathbf{P}}' + \sum_{\alpha \in \mathcal{A}} s_\alpha M_0 \delta^{m-|\alpha|} \mathbf{P}'_\alpha + \sum_{\alpha \in \hat{\mathcal{A}}} t_\alpha M_0 \delta^{m-|\alpha|} \hat{\mathbf{P}}'_\alpha \in \Gamma_{l_0-1}(y_0, CM_0),$$

whenever  $|s_\alpha| \leq c_2$  for all  $\alpha \in \mathcal{A}$  and  $|t_\alpha| \leq c_2$  for all  $\alpha \in \hat{\mathcal{A}}$ .

Note also that (21) may be written in the equivalent form

(27)

$$\begin{aligned} \hat{\mathbf{p}}' &= \mathbf{p}^0 + \left[ \frac{\#(\hat{\mathcal{A}})}{\#(\mathcal{A}) + \#(\hat{\mathcal{A}})} \right] (\hat{\mathbf{p}}^0 - \mathbf{p}^0) \\ &+ \frac{1}{2[\#(\mathcal{A}) + \#(\hat{\mathcal{A}})]} \left\{ \sum_{\alpha \in \mathcal{A}, \sigma = \pm 1} c_0 \sigma M_0 \delta^{m-|\alpha|} \mathbf{E}_{\alpha, \sigma} + \sum_{\alpha \in \hat{\mathcal{A}}, \sigma = \pm 1} \hat{c}_0 \sigma M_0 \delta^{m-|\alpha|} \hat{\mathbf{E}}_{\alpha, \sigma} \right\}. \end{aligned}$$

Consequently, (3), (4), (18), (20) tell us that

$$(28) \quad |\partial^\beta (\hat{\mathbf{p}}' - \mathbf{p}^0)(\mathbf{x}_0)| \leq C \epsilon_0 M_0 \delta^{m-|\beta|} \text{ for } \beta \in \mathcal{A};$$

$$(29) \quad |\partial^\beta (\hat{\mathbf{p}}' - \mathbf{p}^0)(\mathbf{x}_0)| \leq C M_0 \delta^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

Similarly, (13)  $\cdots$  (16), (18), (20), and (22), (23) together imply the estimates

$$(30) \quad |\partial^\beta \mathbf{p}'_\alpha(\mathbf{x}_0) - \delta_{\beta\alpha}| \leq C \epsilon_0 \delta^{|\alpha|-|\beta|} \text{ for } \alpha, \beta \in \mathcal{A};$$

$$(31) \quad |\partial^\beta \hat{\mathbf{p}}'_\alpha(\mathbf{x}_0) - \delta_{\beta\alpha}| \leq C \epsilon_0 \delta^{|\alpha|-|\beta|} \text{ for } \alpha, \beta \in \hat{\mathcal{A}};$$

$$(32) \quad |\partial^\beta \mathbf{p}'_\alpha(\mathbf{x}_0)| \leq C \delta^{|\alpha|-|\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}; \text{ and}$$

$$(33) \quad |\partial^\beta \hat{\mathbf{p}}'_\alpha(\mathbf{x}_0)| \leq C \delta^{|\alpha|-|\beta|} \text{ for } \alpha \in \hat{\mathcal{A}}, \beta \in \mathcal{M}.$$

From (30)  $\cdots$  (33) and (5), we have also

$$(34) \quad |\partial^\beta \mathbf{p}'_\alpha(\mathbf{y}_0) - \delta_{\beta\alpha}| \leq C \epsilon_0 \delta^{|\alpha|-|\beta|} \text{ for } \beta, \alpha \in \mathcal{A};$$

$$(35) \quad |\partial^\beta \hat{\mathbf{p}}'_\alpha(\mathbf{y}_0) - \delta_{\beta\alpha}| \leq C \epsilon_0 \delta^{|\alpha|-|\beta|} \text{ for } \beta, \alpha \in \hat{\mathcal{A}};$$

$$(36) \quad |\partial^\beta \mathbf{p}'_\alpha(\mathbf{y}_0)| \leq C \delta^{|\alpha|-|\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}; \text{ and}$$

$$(37) \quad |\partial^\beta \hat{\mathbf{p}}'_\alpha(\mathbf{y}_0)| \leq C \delta^{|\alpha|-|\beta|} \text{ for } \alpha \in \hat{\mathcal{A}}, \beta \in \mathcal{M}.$$

Next, we prove that there exists  $\hat{\mathbf{p}}^\# \in \mathcal{P}$  with the following properties:

$$(38) \quad \partial^\beta (\hat{\mathbf{p}}^\# - \mathbf{p}^0)(\mathbf{x}_0) = 0 \text{ for } \beta \in \mathcal{A};$$

$$(39) \quad |\partial^\beta (\hat{\mathbf{p}}^\# - \mathbf{p}^0)(\mathbf{x}_0)| \leq C M_0 \delta^{m-|\beta|} \text{ for } \beta \in \mathcal{M};$$

for a small enough controlled constant  $c_3$ , we have

$$(40) \quad \hat{\mathbf{p}}^\# + \sum_{\alpha \in \mathcal{A}} s_\alpha M_0 \delta^{m-|\alpha|} \mathbf{p}'_\alpha + \sum_{\alpha \in \hat{\mathcal{A}}} t_\alpha M_0 \delta^{m-|\alpha|} \hat{\mathbf{p}}'_\alpha \in \Gamma_{l_0-1}(\mathbf{y}_0, C M_0),$$

whenever all  $|s_\alpha|, |t_\alpha|$  are less than  $c_3$ .

Indeed, if  $\mathcal{A} = \emptyset$ , we set  $\hat{\mathbf{p}}^\# = \hat{\mathbf{p}}'$ ; then (38) holds vacuously, and (39), (40) simply restate (29), (26). Suppose  $\mathcal{A} \neq \emptyset$ . We will pick coefficients  $s_\alpha^\#$  ( $\alpha \in \mathcal{A}$ ) for which

$$(41) \quad \hat{\mathbf{p}}^\# := \hat{\mathbf{p}}' + \sum_{\alpha \in \mathcal{A}} s_\alpha^\# \mathbf{M}_0 \delta^{m-|\alpha|} \mathbf{p}'_\alpha \text{ satisfies (38), (39), (40).}$$

In fact, with  $\hat{\mathbf{p}}^\#$  given by (41), equation (38) is equivalent to the system of the linear equations

$$(42) \quad \sum_{\alpha \in \mathcal{A}} \left[ \delta^{|\beta|-|\alpha|} \partial^\beta \mathbf{p}'_\alpha(x_0) \right] s_\alpha^\# = -\mathbf{M}_0^{-1} \delta^{|\beta|-m} \partial^\beta (\hat{\mathbf{p}}' - \mathbf{p}^0)(x_0) \quad (\beta \in \mathcal{A}).$$

By (28), the right-hand side of (42) has absolute value at most  $C\epsilon_0$ . Hence, by (30) and the *small  $\epsilon_0$  assumption* (10), we can solve (42) for the  $s_\alpha^\#$ , and we have

$$(43) \quad |s_\alpha^\#| \leq C\epsilon_0 \text{ for all } \alpha \in \mathcal{A}.$$

The resulting  $\hat{\mathbf{p}}^\#$  given by (41) then satisfies (38). Moreover, for  $\beta \in \mathcal{M}$ , we have

$$\begin{aligned} |\partial^\beta (\hat{\mathbf{p}}^\# - \mathbf{p}^0)(x_0)| &\leq |\partial^\beta (\hat{\mathbf{p}}' - \mathbf{p}^0)(x_0)| + \sum_{\alpha \in \mathcal{A}} |s_\alpha^\#| \cdot \mathbf{M}_0 \delta^{m-|\alpha|} |\partial^\beta \mathbf{p}'_\alpha(x_0)| \\ &\leq |\partial^\beta (\hat{\mathbf{p}}' - \mathbf{p}^0)(x_0)| + C\epsilon_0 \delta^{m-|\beta|} \mathbf{M}_0, \end{aligned}$$

thanks to (32), (41), (43).

Therefore, (29) gives

$$|\partial^\beta (\hat{\mathbf{p}}^\# - \mathbf{p}^0)(x_0)| \leq C\mathbf{M}_0 \delta^{m-|\beta|} \text{ for } \beta \in \mathcal{M},$$

proving (39).

Finally, (40) follows at once from (26), (41), (43) and the *small  $\epsilon_0$  assumption* (10).

Thus, in all cases, there exists  $\hat{\mathbf{p}}^\#$  satisfying (38), (39), (40). We fix such a  $\hat{\mathbf{p}}^\#$ .

Next, we produce an  $(\mathcal{A}, \delta, C)$ -basis for  $\vec{\Gamma}_{l_0-1}$  at  $(y_0, \mathbf{M}_0, \hat{\mathbf{p}}^\#)$ . To do so, we first suppose that  $\mathcal{A} \neq \emptyset$ , and set

$$(44) \quad \mathbf{p}_\gamma^\# = \sum_{\alpha \in \mathcal{A}} \mathbf{b}_{\gamma\alpha} \delta^{|\gamma|-|\alpha|} \mathbf{p}'_\alpha$$

for real coefficients  $(\mathbf{b}_{\gamma\alpha})_{\gamma, \alpha \in \mathcal{A}}$  to be picked below. For  $\beta, \gamma \in \mathcal{A}$ , we have

$$\partial^\beta \mathbf{p}_\gamma^\#(y_0) = \delta^{|\beta|-|\gamma|} \cdot \sum_{\alpha \in \mathcal{A}} \mathbf{b}_{\gamma\alpha} \left[ \delta^{|\beta|-|\alpha|} \partial^\beta \mathbf{p}'_\alpha(y_0) \right].$$

Thanks to (34) and the *small  $\epsilon_0$  assumption* (10), we may define  $(\mathbf{b}_{\gamma\alpha})_{\gamma, \alpha \in \mathcal{A}}$  as the inverse matrix of  $\left( \delta^{|\beta|-|\alpha|} \partial^\beta \mathbf{p}'_\alpha(y_0) \right)_{\alpha, \beta \in \mathcal{A}}$ , and we then have

$$(45) \quad \partial^\beta \mathbf{p}_\gamma^\#(y_0) = \delta_{\beta\gamma} \quad (\beta, \gamma \in \mathcal{A})$$

and

$$(46) \quad |\mathbf{b}_{\gamma\alpha} - \delta_{\gamma\alpha}| \leq C\epsilon_0 \text{ for } \gamma, \alpha \in \mathcal{A}.$$

In particular,

$$(47) \quad |\mathbf{b}_{\gamma\alpha}| \leq C,$$

and therefore for  $\gamma \in \mathcal{A}$ ,  $\beta \in \mathcal{M}$  we have

$$(48)$$

$$\begin{aligned} |\partial^\beta \mathbf{p}_\gamma^\#(y_0)| &\leq \sum_{\alpha \in \mathcal{A}} |\mathbf{b}_{\gamma\alpha}| \delta^{|\gamma| - |\alpha|} |\partial^\beta \mathbf{p}'_\alpha(y_0)| \\ &\leq C \sum_{\alpha \in \mathcal{A}} \delta^{|\gamma| - |\alpha|} \delta^{|\alpha| - |\beta|} \leq C' \delta^{|\gamma| - |\beta|}, \end{aligned}$$

thanks to (36).

Also, for  $\gamma \in \mathcal{A}$ , we have

$$\left[ M_0 \delta^{m - |\gamma|} \mathbf{p}_\gamma^\# \right] = \sum_{\alpha \in \mathcal{A}} \mathbf{b}_{\gamma\alpha} \cdot \left[ M_0 \delta^{m - |\alpha|} \mathbf{p}'_\alpha \right].$$

Therefore, for a small enough controlled constant  $c_4$ , we have

$$(49) \quad \hat{\mathbf{p}}^\# + c_4 M_0 \delta^{m - |\gamma|} \mathbf{p}_\gamma^\#, \hat{\mathbf{p}}^\# - c_4 M_0 \delta^{m - |\gamma|} \mathbf{p}_\gamma^\# \in \Gamma_{l_0 - 1}(y_0, CM_0) \text{ for } \gamma \in \mathcal{A},$$

thanks to (40), which in turn holds thanks to (47). Since we are assuming that  $\mathcal{A} \neq \emptyset$ , (49) implies that also

$$(50) \quad \hat{\mathbf{p}}^\# \in \Gamma_{l_0 - 1}(y_0, CM_0).$$

Our results (45), (48), (49), (50) tell us that  $(\mathbf{p}_\gamma^\#)_{\gamma \in \mathcal{A}}$  is an  $(\mathcal{A}, \delta, C)$ -basis for  $\vec{\Gamma}_{l_0 - 1}$  at  $(y_0, M_0, \hat{\mathbf{p}}^\#)$ .

Thus, we have produced the desired  $(\mathcal{A}, \delta, C)$ -basis, provided that  $\mathcal{A} \neq \emptyset$ . On the other hand, if  $\mathcal{A} = \emptyset$ , then the existence of an  $(\mathcal{A}, \delta, C)$ -basis for  $\vec{\Gamma}_{l_0 - 1}$  at  $(y_0, M_0, \hat{\mathbf{p}}^\#)$  is equivalent to the assertion that

$$(51) \quad \hat{\mathbf{p}}^\# \in \Gamma_{l_0 - 1}(y_0, CM_0),$$

and (51) follows at once from (40). Thus, in all cases,

$$(52) \quad \vec{\Gamma}_{l_0 - 1} \text{ has an } (\mathcal{A}, \delta, C)\text{-basis at } (y_0, M_0, \hat{\mathbf{p}}^\#).$$

Similarly, we can produce an  $(\hat{\mathcal{A}}, \delta, C)$ -basis for  $\vec{\Gamma}_{l_0 - 1}$  at  $(y_0, M_0, \hat{\mathbf{p}}^\#)$ . We suppose first that  $\hat{\mathcal{A}} \neq \emptyset$ , and set

$$(53) \quad \hat{\mathbf{p}}_\gamma^\# = \sum_{\alpha \in \hat{\mathcal{A}}} \hat{\mathbf{b}}_{\beta\alpha} \delta^{|\gamma| - |\alpha|} \hat{\mathbf{p}}'_\alpha \text{ for } \gamma \in \hat{\mathcal{A}}, \text{ with coefficients } \hat{\mathbf{b}}_{\gamma\alpha} \text{ to be picked below.}$$

Thanks to (35) and the *small  $\epsilon_0$  assumption* (10), we can pick the coefficients  $\hat{\mathbf{b}}_{\gamma\alpha}$  so that

$$(54) \quad \partial^\beta \hat{\mathbf{p}}_\gamma^\#(\mathbf{y}_0) = \delta_{\beta,\gamma} \text{ for } \beta, \gamma \in \hat{\mathcal{A}}$$

and

$$(55) \quad |\hat{\mathbf{b}}_{\gamma\alpha} - \delta_{\gamma\alpha}| \leq C\epsilon_0 \text{ for } \gamma, \alpha \in \hat{\mathcal{A}},$$

hence

$$(56) \quad |\hat{\mathbf{b}}_{\gamma\alpha}| \leq C \text{ for } \gamma, \alpha \in \hat{\mathcal{A}}.$$

From (37), (53), (56), we obtain the estimate

$$(57) \quad \left| \partial^\beta \hat{\mathbf{p}}_\gamma^\#(\mathbf{y}_0) \right| \leq C\delta^{|\gamma|-|\beta|} \text{ for } \gamma \in \hat{\mathcal{A}} \text{ and } \beta \in \mathcal{M},$$

in analogy with (48). Also for  $\gamma \in \hat{\mathcal{A}}$ , we have

$$\mathbf{M}_0 \delta^{m-|\gamma|} \hat{\mathbf{p}}_\gamma^\# = \sum_{\alpha \in \hat{\mathcal{A}}} \hat{\mathbf{b}}_{\gamma\alpha} \left[ \mathbf{M}_0 \delta^{m-|\alpha|} \hat{\mathbf{p}}_\alpha' \right].$$

Together with (40) and (56), this tells us that

$$(58) \quad \hat{\mathbf{p}}^\# + c_5 \mathbf{M}_0 \delta^{m-|\gamma|} \hat{\mathbf{p}}_\gamma^\#, \hat{\mathbf{p}}^\# - c_5 \mathbf{M}_0 \delta^{m-|\gamma|} \hat{\mathbf{p}}_\gamma^\# \in \Gamma_{l_0-1}(\mathbf{y}_0, C\mathbf{M}_0) \text{ for } \gamma \in \hat{\mathcal{A}}$$

in analogy with (49). Since we are assuming that  $\hat{\mathcal{A}} \neq \emptyset$ , (58) implies that

$$(59) \quad \hat{\mathbf{p}}^\# \in \Gamma_{l_0-1}(\mathbf{y}_0, C\mathbf{M}_0).$$

Our results (54), (57), (58), (59) tell us that  $\left( \hat{\mathbf{p}}_\gamma^\# \right)_{\gamma \in \hat{\mathcal{A}}}$  is an  $(\hat{\mathcal{A}}, \delta, C)$ -basis for  $\vec{\Gamma}_{l_0-1}$  at  $(\mathbf{y}_0, \mathbf{M}_0, \hat{\mathbf{p}}^\#)$ .

Thus, we have produced the desired  $(\hat{\mathcal{A}}, \delta, C)$ -basis, provided  $\hat{\mathcal{A}} \neq \emptyset$ . On the other hand, if  $\hat{\mathcal{A}} = \emptyset$ , then the existence of an  $(\hat{\mathcal{A}}, \delta, C)$ -basis for  $\vec{\Gamma}_{l_0-1}$  at  $(\mathbf{y}_0, \mathbf{M}_0, \hat{\mathbf{p}}^\#)$  is equivalent to the assertion that

$$(60) \quad \hat{\mathbf{p}}^\# \in \Gamma_{l_0-1}(\mathbf{y}_0, C\mathbf{M}_0),$$

and (60) follows at once from (40). Thus, in all cases,

$$(61) \quad \vec{\Gamma}_{l_0-1} \text{ has an } (\hat{\mathcal{A}}, \delta, C)\text{-basis at } (\mathbf{y}_0, \mathbf{M}_0, \hat{\mathbf{p}}^\#).$$

Our results (52) and (61) together yield conclusion (6) of the Transport Lemma (Lemma 9). Also, our results (38) and (39) imply conclusions (7) and (8), since  $\mathcal{A}$  is monotonic. (See (1).)

Thus, starting from assumptions (1)  $\cdots$  (5) and (9), (10), we have proven conclusions (6), (7), (8) for our  $\hat{\mathbf{p}}^\#$ .

The proof of the Transport Lemma (Lemma 9) is complete. ■

**Remark** *The monotonicity of  $\mathcal{A}$  was used in the proof of the Transport Lemma only to guarantee that formulas (3) and (7) are independent of the base point.*

## Part II

# The Main Lemma

### II.1 Statement of the Main Lemma

For  $\mathcal{A} \subseteq \mathcal{M}$  monotonic, we define

$$(1) \quad \mathfrak{l}(\mathcal{A}) = 1 + 3 \cdot \#\{\mathcal{A}' \subseteq \mathcal{M} : \mathcal{A}' \text{ monotonic, } \mathcal{A}' < \mathcal{A}\}.$$

Thus,

$$(2) \quad \mathfrak{l}(\mathcal{A}) - 3 \geq \mathfrak{l}(\mathcal{A}') \text{ for } \mathcal{A}', \mathcal{A} \subseteq \mathcal{M} \text{ monotonic with } \mathcal{A}' < \mathcal{A}.$$

By induction on  $\mathcal{A}$  (with respect to the order relation  $<$ ), we will prove the following result.

**Main Lemma for  $\mathcal{A}$**  *Let  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field, and for  $\mathfrak{l} \geq 1$ , let  $\vec{\Gamma}_{\mathfrak{l}} = (\Gamma_{\mathfrak{l}}(x, M))_{x \in E, M > 0}$  be the  $\mathfrak{l}$ -th refinement of  $\vec{\Gamma}_0$ . Fix a dyadic cube  $Q_0 \subset \mathbb{R}^n$ , a point  $x_0 \in E \cap 5(Q_0^+)$  and a polynomial  $P^0 \in \mathcal{P}$ , as well as positive real numbers  $M_0, \epsilon, C_B$ . We make the following assumptions.*

$$(A1) \quad \vec{\Gamma}_{\mathfrak{l}(\mathcal{A})} \text{ has an } (\mathcal{A}, \epsilon^{-1} \delta_{Q_0}, C_B)\text{-basis at } (x_0, M_0, P^0).$$

$$(A2) \quad \epsilon^{-1} \delta_{Q_0} \leq \delta_{\max}.$$

$$(A3) \quad (\text{“Small } \epsilon \text{ Assumption”}) \quad \epsilon \text{ is less than a small enough constant determined by } C_B, C_w, m, n.$$

*Then there exists  $F \in C^m(\frac{65}{64}Q_0)$  satisfying the following conditions.*

$$(C1) \quad |\partial^\beta (F - P^0)| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|} \text{ on } \frac{65}{64}Q_0 \text{ for } |\beta| \leq m, \text{ where } C(\epsilon) \text{ is determined by } \epsilon, C_B, C_w, m, n.$$

$$(C2) \quad J_z(F) \in \Gamma_0(z, C'(\epsilon) M_0) \text{ for all } z \in E \cap \frac{65}{64}Q_0, \text{ where } C'(\epsilon) \text{ is determined by } \epsilon, C_B, C_w, m, n.$$

**Remarks** • *We state the Main Lemma only for monotonic  $\mathcal{A}$ .*

- *Note that  $x_0$  may fail to belong to  $\frac{65}{64}Q_0$ , hence the assertion  $J_{x_0}(F) = P^0$  may be meaningless. Even if  $x_0 \in \frac{65}{64}Q_0$ , we do not assert that  $J_{x_0}(F) = P^0$ . However, see Corollary 4 in Section III.1 below.*

## II.2 The Base Case

The base case of our induction on  $\mathcal{A}$  is the case  $\mathcal{A} = \mathcal{M}$ .

In this section, we prove the Main Lemma for  $\mathcal{M}$ . The hypotheses of the lemma are as follows:

- (1)  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  is a  $(C_w, \delta_{\max})$ -convex shape field.
- (2)  $\vec{\Gamma}_1 = (\Gamma_1(x, M))_{x \in E, M > 0}$  is the first refinement of  $\vec{\Gamma}_0$ .
- (3)  $\vec{\Gamma}_1$  has an  $(\mathcal{M}, \epsilon^{-1}\delta_{Q_0}, C_B)$ -basis at  $(x_0, M_0, P^0)$ .
- (4)  $\epsilon^{-1}\delta_{Q_0} \leq \delta_{\max}$ .
- (5)  $\epsilon$  is less than a small enough constant determined by  $C_B, C_w, m, n$ .
- (6)  $x_0 \in 5(Q_0)^+ \cap E$ .

We write  $c, C, C'$ , etc., to denote constants determined by  $C_B, C_w, m, n$ . These symbols may denote different constants in different occurrences.

- (7) Let  $z \in E \cap \frac{65}{64}Q_0$ .

Then (6), (7) imply that

- (8)  $|z - x_0| \leq C\delta_{Q_0} = C\epsilon \cdot (\epsilon^{-1}\delta_{Q_0})$ .

From (1), (2), (3), (5), (8), and Lemma 9 in Section I.4 (with  $\hat{\mathcal{A}} = \mathcal{A}$ ,  $\hat{P}^0 = P^0$ ), we obtain a polynomial  $\hat{P}^\# \in \mathcal{P}$  such that

- (9)  $\vec{\Gamma}_0$  has an  $(\mathcal{M}, \epsilon^{-1}\delta_{Q_0}, C')$ -basis at  $(z, M_0, \hat{P}^\#)$ , and

- (10)  $\partial^\beta (\hat{P}^\# - P^0) = 0$  for  $\beta \in \mathcal{M}$ .

From (9), we have  $\hat{P}^\# \in \Gamma_0(z, C'M_0)$ , while (10) tells us that  $\hat{P}^\# = P^0$ . Thus,

- (11)  $P^0 \in \Gamma_0(z, C'M_0)$  for all  $z \in \frac{65}{64}Q_0 \cap E$ .

Consequently, the function  $F := P^0$  on  $\frac{65}{64}Q_0$  satisfies the conclusions (C1), (C2) of the Main Lemma for  $\mathcal{M}$ .

This completes the proof of the Main Lemma for  $\mathcal{M}$ . ■

## II.3 Setup for the Induction Step

Fix a monotonic set  $\mathcal{A}$  strictly contained in  $\mathcal{M}$ , and assume the following

- (1) Induction Hypothesis: The Main Lemma for  $\mathcal{A}'$  holds for all monotonic  $\mathcal{A}' < \mathcal{A}$ .

Under this assumption, we will prove the Main Lemma for  $\mathcal{A}$ . Thus, let  $\vec{\Gamma}_0$ ,  $\vec{\Gamma}_l$  ( $l \geq 1$ ),  $C_w$ ,  $\delta_{\max}$ ,  $Q_0$ ,  $x_0$ ,  $P^0$ ,  $M_0$ ,  $\epsilon$ ,  $C_B$  be as in the hypotheses of the Main Lemma for  $\mathcal{A}$ . Our goal is to prove the existence of  $F \in C^m(\frac{65}{64}Q_0)$  satisfying conditions (C1) and (C2). To do so, we introduce a constant  $A \geq 1$ , and make the following additional assumptions.

- (2) Large  $A$  assumption:  $A$  exceeds a large enough constant determined by  $C_B$ ,  $C_w$ ,  $m$ ,  $n$ .
- (3) Small  $\epsilon$  assumption:  $\epsilon$  is less than a small enough constant determined by  $A$ ,  $C_B$ ,  $C_w$ ,  $m$ ,  $n$ .

We write  $c$ ,  $C$ ,  $C'$ , etc., to denote constants determined by  $C_B$ ,  $C_w$ ,  $m$ ,  $n$ . Also we write  $c(A)$ ,  $C(A)$ ,  $C'(A)$ , etc., to denote constants determined by  $A$ ,  $C_B$ ,  $C_w$ ,  $m$ ,  $n$ . Similarly, we write  $C(\epsilon)$ ,  $c(\epsilon)$ ,  $C'(\epsilon)$ , etc., to denote constants determined by  $\epsilon$ ,  $A$ ,  $C_B$ ,  $C_w$ ,  $m$ ,  $n$ . These symbols may denote different constants in different occurrences.

In place of (C1), (C2), we will prove the existence of a function  $F \in C^m(\frac{65}{64}Q_0)$  satisfying

$$(C^*1) \quad |\partial^\beta (F - P^0)| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|} \text{ on } \frac{65}{64}Q_0 \text{ for } |\beta| \leq \mathcal{M}; \text{ and}$$

$$(C^*2) \quad J_z(F) \in \Gamma_0(z, C(\epsilon) M_0) \text{ for all } z \in E \cap \frac{65}{64}Q_0.$$

Conditions (C\*1), (C\*2) differ from (C1), (C2) in that the constants in (C\*1), (C\*2) may depend on  $A$ .

Once we establish (C\*1) and (C\*2), we may fix  $A$  to be a constant determined by  $C_B$ ,  $C_w$ ,  $m$ ,  $n$ , large enough to satisfy the Large  $A$  Assumption (2). The Small  $\epsilon$  Assumption (3) will then follow from the Small  $\epsilon$  Assumption (A3) in the Main Lemma for  $\mathcal{A}$ ; and the desired conclusions (C1), (C2) will then follow from (C\*1), (C\*2).

Thus, our goal is to prove the existence of  $F \in C^m(\frac{65}{64}Q_0)$  satisfying (C\*1) and (C\*2), assuming (1), (2), (3) above, along with hypotheses of the Main Lemma for  $\mathcal{A}$ . This will complete our induction on  $\mathcal{A}$  and establish the Main Lemma for all monotonic subsets of  $\mathcal{M}$ .

## II.4 Calderón-Zygmund Decomposition

We place ourselves in the setting of Section II.3. Let  $Q$  be a dyadic cube. We say that  $Q$  is “OK” if (1) and (2) below are satisfied.



- (1)  $5Q \subseteq 5Q_0$ .
- (2) Either  $\#(E \cap 5Q) \leq 1$  or there exists  $\hat{A} < \mathcal{A}$  (strict inequality) for which the following holds:
- (3) For each  $y \in E \cap 5Q$  there exists  $\hat{P}^y \in \mathcal{P}$  satisfying
  - (3a)  $\vec{\Gamma}_{l(\mathcal{A})-3}$  has a weak  $(\hat{A}, \epsilon^{-1}\delta_Q, A)$ -basis at  $(y, M_0, \hat{P}^y)$ .
  - (3b)  $|\partial^\beta (\hat{P}^y - P^0)(x_0)| \leq AM_0 (\epsilon^{-1}\delta_{Q_0})^{m-|\beta|}$  for all  $\beta \in \mathcal{M}$ .
  - (3c)  $\partial^\beta (\hat{P}^y - P^0) \equiv 0$  for  $\beta \in \mathcal{A}$ .

**Remark** *The argument in this section and the next will depend sensitively on several details of the above definition. Note that (3a) involves  $\vec{\Gamma}_{l(\mathcal{A})-3}$  rather than  $\vec{\Gamma}_{l(\hat{A})}$ , and that (3b) involves  $x_0, \delta_{Q_0}$  rather than  $y, \delta_Q$ . Note also that the set  $\hat{A}$  in (2), (3) needn't be monotonic.*

A dyadic cube  $Q$  will be called a Calderón-Zygmund cube (or a CZ cube) if it is OK, but no dyadic cube strictly containing  $Q$  is OK.

Recall that given any two distinct dyadic cubes  $Q, Q'$ , either  $Q$  is strictly contained in  $Q'$ , or  $Q'$  is strictly contained in  $Q$ , or  $Q \cap Q' = \emptyset$ . The first two alternatives here are ruled out if  $Q, Q'$  are CZ cubes. Hence, the Calderón-Zygmund cubes are pairwise disjoint.

Any CZ cube  $Q$  satisfies (1) and is therefore contained in the interior of  $5Q_0$ . On the other hand, let  $x$  be an interior point of  $5Q_0$ . Then any sufficiently small dyadic cube  $Q$  containing  $x$  satisfies  $5Q \subset 5Q_0$  and  $\#(E \cap 5Q) \leq 1$ ; such  $Q$  are OK. However, any sufficiently large dyadic cube  $Q$  containing  $x$  will fail to satisfy  $5Q \subseteq 5Q_0$ ; such  $Q$  are not OK. It follows that  $x$  is contained in a maximal OK dyadic cube. Thus, we have proven

**Lemma 10** *The CZ cubes form a partition of the interior of  $5Q_0$ .*

Next, we establish

**Lemma 11** *Let  $Q, Q'$  be CZ cubes. If  $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$ , then  $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$ .*

**Proof.** Suppose not. Without loss of generality, we may suppose that  $\delta_Q \leq \frac{1}{4}\delta_{Q'}$ . Then  $\delta_{Q^+} \leq \frac{1}{2}\delta_{Q'}$ , and  $\frac{65}{64}Q^+ \cap \frac{65}{64}Q' \neq \emptyset$ ; hence,  $5Q^+ \subset 5Q'$ . The cube  $Q'$  is OK. Therefore,

$$(4) \quad 5Q^+ \subset 5Q' \subseteq 5Q_0.$$

If  $\#(E \cap 5Q') \leq 1$ , then also  $\#(E \cap 5Q^+) \leq 1$ . Otherwise, there exists  $\hat{A} < \mathcal{A}$  such that for each  $y \in E \cap 5Q'$  there exists  $\hat{P}^y \in \mathcal{P}$  satisfying

- (5)  $\vec{\Gamma}_{l(\mathcal{A})-3}$  has a weak  $(\hat{A}, \epsilon^{-1}\delta_{Q'}, A)$ -basis at  $(y, M_0, \hat{P}^y)$ ,
- (6)  $|\partial^\beta (\hat{P}^y - P^0)(x_0)| \leq AM_0 (\epsilon^{-1}\delta_{Q_0})^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ , and

$$(7) \quad \partial^\beta (\hat{P}^y - P^0) \equiv 0 \text{ for } \beta \in \mathcal{A}.$$

For each  $y \in E \cap 5Q^+ \subseteq E \cap 5Q'$ , the above  $\hat{P}^y$  satisfies (6), (7); and (5) implies

$$(8) \quad \vec{\Gamma}_{l(\mathcal{A})-3} \text{ has a weak } (\hat{\mathcal{A}}, \epsilon^{-1}\delta_{Q^+}, \mathcal{A})\text{-basis at } (y, M_0, \hat{P}^y)$$

because  $\epsilon^{-1}\delta_{Q^+} < \epsilon^{-1}\delta_{Q'}$ , and because (5), (8) deal with weak bases. (See remarks in Section I.3.) Thus, (4) holds, and either  $\#(E \cap 5Q^+) \leq 1$  or else our  $\hat{\mathcal{A}} < \mathcal{A}$  and  $\hat{P}^y$  ( $y \in E \cap 5Q^+$ ) satisfy (6), (7), (8). This tells us that  $Q^+$  is OK. However,  $Q^+$  strictly contains the CZ cube  $Q$ ; therefore,  $Q^+$  cannot be OK. This contradiction completes the proof of Lemma 11. ■

Note that the proof of Lemma 11 made use of our decision to involve  $x_0$ ,  $\delta_{Q_0}$  rather than  $y$ ,  $\delta_Q$  in (3b), as well as our decision to use weak bases in (3a).

We also have the following easy lemma.

**Lemma 12** *Only finitely many CZ cubes  $Q$  satisfy the condition*

$$(9) \quad \frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset.$$

## II.5 Auxiliary Polynomials

We again place ourselves in the setting of Section II.3 and we make use of the Calderón-Zygmund decomposition defined in Section II.4.

Recall that  $x_0 \in E \cap 5Q_0^+$ , and that  $\vec{\Gamma}_{l(\mathcal{A})}$  has an  $(\mathcal{A}, \epsilon^{-1}\delta_{Q_0}, C_B)$ -basis at  $(x_0, M_0, P^0)$ ; moreover,  $\mathcal{A} \subseteq \mathcal{M}$  is monotonic, and  $\epsilon$  is less than a small enough constant determined by  $C_B, C_w, m, n$ .

Let  $y \in E \cap 5Q_0$ . Then  $|x_0 - y| \leq C\delta_{Q_0} = (C\epsilon)(\epsilon^{-1}\delta_{Q_0})$ . Applying Lemma 9 in Section I.4 with  $\hat{\mathcal{A}} = \mathcal{A}$ ,  $\hat{P}^0 = P^0$ , we see that there exists  $P^y \in \mathcal{P}$  with the following properties.

$$(1) \quad \vec{\Gamma}_{l(\mathcal{A})-1} \text{ has an } (\mathcal{A}, \epsilon^{-1}\delta_{Q_0}, C)\text{-basis } (P_\alpha^y)_{\alpha \in \mathcal{A}} \text{ at } (y, M_0, P^y),$$

$$(2) \quad \partial^\beta (P^y - P^0) \equiv 0 \text{ for } \beta \in \mathcal{A},$$

$$(3) \quad |\partial^\beta (P^y - P^0)(x_0)| \leq CM_0 (\epsilon^{-1}\delta_{Q_0})^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

We fix  $P^y, P_\alpha^y$  ( $\alpha \in \mathcal{A}$ ) as above for each  $y \in E \cap 5Q_0$ . We study the relationship between the polynomials  $P^y, P_\alpha^y$  ( $\alpha \in \mathcal{A}$ ) and the Calderón-Zygmund decomposition.

**Lemma 13 (“Controlled Auxiliary Polynomials”)** *Let  $Q \in CZ$ , and suppose that*

$$(4) \quad \frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset.$$

*Let*

$$(5) \quad y \in E \cap 5Q_0 \cap 5Q^+.$$

Then

$$(6) \quad |\partial^\beta P_\alpha^y(y)| \leq C \cdot (\epsilon^{-1} \delta_Q)^{|\alpha| - |\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}.$$

**Proof.**

If  $\mathcal{A} = \emptyset$ , then (6) holds vacuously. Suppose  $\mathcal{A} \neq \emptyset$ .

Let  $K \geq 1$  be a large enough constant to be picked below, and assume that

$$(7) \quad \max_{\alpha \in \mathcal{A}, \beta \in \mathcal{M}} (\epsilon^{-1} \delta_Q)^{|\beta| - |\alpha|} |\partial^\beta P_\alpha^y(y)| > K.$$

We will derive a contradiction.

Thanks to (1), we have

$$(8) \quad P^y, P^y \pm CM_0 \cdot (\epsilon^{-1} \delta_{Q_0})^{m - |\alpha|} P_\alpha^y \in \Gamma_{l(\mathcal{A}) - 1}(y, CM_0) \text{ for } \alpha \in \mathcal{A},$$

$$(9) \quad \partial^\beta P_\alpha^y(y) = \delta_{\beta\alpha} \text{ for } \beta, \alpha \in \mathcal{A},$$

and

$$(10) \quad |\partial^\beta P_\alpha^y(y)| \leq C (\epsilon^{-1} \delta_{Q_0})^{|\alpha| - |\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}.$$

Also,

$$(11) \quad 5Q \subset 5Q_0 \text{ since } Q \text{ is OK.}$$

If  $\delta_Q \geq 2^{-12} \delta_{Q_0}$ , then from (10), (11), we would have

$$(12) \quad \max_{\alpha \in \mathcal{A}, \beta \in \mathcal{M}} (\epsilon^{-1} \delta_Q)^{|\beta| - |\alpha|} |\partial^\beta P_\alpha^y(y)| \leq C'.$$

We will pick

$$(13) \quad K > C', \text{ with } C' \text{ as in (12).}$$

Then (12) contradicts our assumption (7).

Thus, we must have

$$(14) \quad \delta_Q < 2^{-12} \delta_{Q_0}.$$

Let

$$(15) \quad Q = \hat{Q}_0 \subset \hat{Q}_1 \subset \cdots \subset \hat{Q}_{\nu_{\max}} \text{ be all the dyadic cubes containing } Q \text{ and having sidelength at most } 2^{-10} \delta_{Q_0}.$$

Then

$$(16) \quad \hat{Q}_0 = Q, \delta_{\hat{Q}_{\nu_{\max}}} = 2^{-10} \delta_{Q_0}, \hat{Q}_{\nu+1} = (\hat{Q}_\nu)^+ \text{ for } 0 \leq \nu \leq \nu_{\max} - 1, \text{ and } \nu_{\max} \geq 2.$$

For  $1 \leq \nu \leq \nu_{\max}$ , we define

$$(17) \quad X_\nu = \max_{\alpha \in \mathcal{A}, \beta \in \mathcal{M}} \left( \epsilon^{-1} \delta_{\tilde{Q}_\nu} \right)^{|\beta| - |\alpha|} |\partial^\beta P_\alpha^y(y)|.$$

From (7) and (10), we have

$$(18) \quad X_0 > K, X_{\nu_{\max}} \leq C',$$

and from (16), (17), we have

$$(19) \quad 2^{-m} X_\nu \leq X_{\nu+1} \leq 2^m X_\nu, \text{ for } 0 \leq \nu \leq \nu_{\max}.$$

We will pick

$$(20) \quad K > C' \text{ with } C' \text{ as in (18).}$$

Then  $\tilde{\nu} := \min\{\nu : X_\nu \leq K\}$  and  $\tilde{Q} = \hat{Q}_{\tilde{\nu}}$  satisfy the following, thanks to (18), (19), (20):  $\tilde{\nu} \neq 0$ , hence

$$(21) \quad \tilde{Q} \text{ is a dyadic cube strictly containing } Q; \text{ also } 2^{-m} K \leq X_{\tilde{\nu}} \leq K,$$

hence

$$(22) \quad 2^{-m} K \leq \max_{\alpha \in \mathcal{A}, \beta \in \mathcal{M}} \left( \epsilon^{-1} \delta_{\tilde{Q}} \right)^{|\beta| - |\alpha|} |\partial^\beta P_\alpha^y(y)| \leq K.$$

Also, since  $Q \subset \tilde{Q}$ , we have  $\frac{65}{64} \tilde{Q} \cap \frac{65}{64} Q_0 \neq \emptyset$  by (4); and since  $\delta_{\tilde{Q}} \leq 2^{-10} \delta_{Q_0}$ , we conclude that

$$(23) \quad 5\tilde{Q} \subset 5Q_0.$$

From (8), (10), and (23), we have

$$(24) \quad P^y, P^y \pm cM_0 \left( \epsilon^{-1} \delta_{\tilde{Q}} \right)^{m - |\alpha|} P_\alpha^y \in \Gamma_{l(\mathcal{A})-1}(y, CM_0) \subset \Gamma_{l(\mathcal{A})-2}(y, CM_0) \\ \text{for } \alpha \in \mathcal{A};$$

and

$$(25) \quad |\partial^\beta P_\alpha^y(y)| \leq C \left( \epsilon^{-1} \delta_{\tilde{Q}} \right)^{|\alpha| - |\beta|} \text{ for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}, \beta \geq \alpha.$$

Our results (9), (24), (25) tell us that

$$(26) \quad (P_\alpha^y)_{\alpha \in \mathcal{A}} \text{ is a weak } \left( \mathcal{A}, \epsilon^{-1} \delta_{\tilde{Q}}, C \right)\text{-basis for } \vec{\Gamma}_{l(\mathcal{A})-2} \text{ at } (y, M_0, P^y).$$

Note also that

$$(27) \quad \epsilon^{-1} \delta_{\tilde{Q}} \leq \epsilon^{-1} \delta_{Q_0} \leq \delta_{\max}, \text{ by (23) and hypothesis (A2) of the Main Lemma for } \mathcal{A}.$$

Moreover,

(28)  $\vec{\Gamma}_{l(\mathcal{A})-2}$  is  $(C, \delta_{\max})$ -convex, thanks to Lemma 4 (B).

If we take

(29)  $K \geq C^*$  for a large enough  $C^*$ ,

then (22), (26)  $\dots$  (29) and the Relabeling Lemma (Lemma 7) produce a monotonic set  $\hat{\mathcal{A}} \subset \mathcal{M}$ , such that

(30)  $\hat{\mathcal{A}} < \mathcal{A}$  (strict inequality)

and

(31)  $\vec{\Gamma}_{l(\mathcal{A})-2}$  has an  $(\hat{\mathcal{A}}, \epsilon^{-1} \delta_{\tilde{Q}}, C)$ -basis at  $(y, M_0, P^y)$ .

Also, from (9), (22), (24), we see that

(32)  $(P^y_{\alpha})_{\alpha \in \mathcal{A}}$  is an  $(\mathcal{A}, \epsilon^{-1} \delta_{\tilde{Q}}, CK)$ -basis for  $\vec{\Gamma}_{l(\mathcal{A})-2}$  at  $(y, M_0, P^y)$ .

We now pick

(33)  $K = \hat{C}$  (a constant determined by  $C_B, C_w, m, n$ ), with  $\hat{C} \geq 1$  large enough to satisfy (13), (20), (29).

Then (31) and (32) tell us that

(34)  $\vec{\Gamma}_{l(\mathcal{A})-2}$  has both an  $(\hat{\mathcal{A}}, \epsilon^{-1} \delta_{\tilde{Q}}, C)$ -basis and an  $(\mathcal{A}, \epsilon^{-1} \delta_{\tilde{Q}}, C)$ -basis at  $(y, M_0, P^y)$ .

Let  $z \in E \cap 5\tilde{Q}$ . Then  $z, y \in 5\tilde{Q}^+$  (see (5)), hence

(35)  $|z - y| \leq C\delta_{\tilde{Q}} = C\epsilon \cdot (\epsilon^{-1} \delta_{\tilde{Q}})$ .

From (34), (35), the Small  $\epsilon$  Assumption and Lemma 9 (and our hypothesis that  $\mathcal{A}$  is monotonic; see Section II.3), we obtain a polynomial  $\check{P}^z \in \mathcal{P}$ , such that

(36)  $\vec{\Gamma}_{l(\mathcal{A})-3}$  has an  $(\hat{\mathcal{A}}, \epsilon^{-1} \delta_{\tilde{Q}}, C)$ -basis at  $(z, M_0, \check{P}^z)$ ,

(37)  $\partial^\beta (\check{P}^z - P^y) \equiv 0$  for  $\beta \in \mathcal{A}$ ,

and

(38)  $|\partial^\beta (\check{P}^z - P^y)(y)| \leq CM_0 \left( \epsilon^{-1} \delta_{\tilde{Q}} \right)^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ .

From (23) and (38), we have

$$(39) \quad |\partial^\beta (\check{P}^z - P^y)(y)| \leq CM_0 (\epsilon^{-1} \delta_{Q_0})^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

Since  $y \in 5Q_0$  by hypothesis of Lemma 13, while  $x_0 \in 5Q_0^+$  by hypothesis of the Main Lemma for  $\mathcal{A}$ , we have  $|x_0 - y| \leq C\delta_{Q_0}$ , and therefore (39) implies that

$$(40) \quad |\partial^\beta (\check{P}^z - P^y)(x_0)| \leq CM_0 (\epsilon^{-1} \delta_{Q_0})^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

From (2), (3), (37), (40), we now have

$$(41) \quad \partial^\beta (\check{P}^z - P^0) \equiv 0 \text{ for } \beta \in \mathcal{A}$$

and

$$(42) \quad |\partial^\beta (\check{P}^z - P^0)(x_0)| \leq CM_0 (\epsilon^{-1} \delta_{Q_0})^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

Our results (36), (41), (42) hold for every  $z \in E \cap 5\tilde{Q}$ .

We recall the Large  $\mathcal{A}$  Assumption in the Section II.3. Then (23), (30), (36), (41), (42) yield the following results:  $5\tilde{Q} \subset 5Q_0$ ,  $\hat{\mathcal{A}} < \mathcal{A}$  (strict inequality).

For every  $z \in E \cap 5\tilde{Q}$ , there exists  $\check{P}^z \in \mathcal{P}$  such that

- $\vec{\Gamma}_{l(\mathcal{A})-3}$  has an  $(\hat{\mathcal{A}}, \epsilon^{-1} \delta_{\tilde{Q}}, \mathcal{A})$ -basis at  $(z, M_0, \check{P}^z)$ .
- $\partial^\beta (\check{P}^z - P^0) \equiv 0$  for  $\beta \in \mathcal{A}$ .
- $|\partial^\beta (\check{P}^z - P^0)(x_0)| \leq AM_0 (\epsilon^{-1} \delta_{Q_0})^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ .

Comparing the above results with the definition of an OK cube, we conclude that  $\tilde{Q}$  is OK.

However, since  $\tilde{Q}$  properly contains the CZ cube  $Q$ , (see (21)),  $\tilde{Q}$  cannot be OK.

This contradiction proves that our assumption (7) must be false.

Thus,  $|\partial^\beta P_\alpha^y(y)| \leq K (\epsilon^{-1} \delta_Q)^{|\alpha|-|\beta|}$  for  $\alpha \in \mathcal{A}$ ,  $\beta \in \mathcal{M}$ .

Since we picked  $K = \hat{C}$  in (33), this implies the estimate (6), completing the proof of Lemma 13. ■

As an easy consequence of Lemma 13, we have the following result.

**Corollary 1** *Let  $Q \in CZ$ , and suppose  $\frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset$ . Let  $y \in E \cap 5Q_0 \cap 5Q^+$ . Then  $(P_\alpha^y)_{\alpha \in \mathcal{A}}$  is an  $(\mathcal{A}, \epsilon^{-1} \delta_Q, C)$ -basis for  $\vec{\Gamma}_{l(\mathcal{A})-1}$  at  $(y, M_0, P^y)$ .*

**Lemma 14 (“Consistency of Auxiliary Polynomials”)** *Let  $Q, Q' \in CZ$ , with*

$$(43) \quad \frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset, \quad \frac{65}{64}Q' \cap \frac{65}{64}Q_0 \neq \emptyset$$

and

$$(44) \quad \frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset.$$

Let

$$(45) \quad \mathbf{y} \in \mathbb{E} \cap 5Q_0 \cap 5Q^+, \mathbf{y}' \in \mathbb{E} \cap 5Q_0 \cap 5(Q')^+.$$

Then

$$(46) \quad \left| \partial^\beta \left( \mathbf{p}^{\mathbf{y}} - \mathbf{p}^{\mathbf{y}'} \right) (\mathbf{y}') \right| \leq CM_0 (\epsilon^{-1} \delta_Q)^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

**Proof.** Suppose first that  $\delta_Q \geq 2^{-20} \delta_{Q_0}$ . Then (3) (applied to  $\mathbf{y}$  and to  $\mathbf{y}'$ ) tells us that

$$\left| \partial^\beta \left( \mathbf{p}^{\mathbf{y}} - \mathbf{p}^{\mathbf{y}'} \right) (\mathbf{x}_0) \right| \leq CM_0 (\epsilon^{-1} \delta_{Q_0})^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

Hence,  $\left| \partial^\beta \left( \mathbf{p}^{\mathbf{y}} - \mathbf{p}^{\mathbf{y}'} \right) (\mathbf{y}') \right| \leq C'M_0 (\epsilon^{-1} \delta_{Q_0})^{m-|\beta|} \leq C''M_0 (\epsilon^{-1} \delta_Q)^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ , since  $\mathbf{x}_0, \mathbf{y}' \in 5Q_0^+$ . Thus, (46) holds if  $\delta_Q \geq 2^{-20} \delta_{Q_0}$ . Suppose

$$(47) \quad \delta_Q < 2^{-20} \delta_{Q_0}.$$

By (44) and Lemma 11, we have

$$(48) \quad \delta_Q, \delta_{Q'} \leq 2^{-20} \delta_{Q_0} \text{ and } \frac{1}{2} \delta_Q \leq \delta_{Q'} \leq 2\delta_Q.$$

Together with (43), this implies that

$$(49) \quad 5Q^+, 5(Q')^+ \subseteq 5Q_0.$$

From Corollary 1, we have

$$(50) \quad \vec{\Gamma}_{l(\mathcal{A})-1} \text{ has an } (\mathcal{A}, \epsilon^{-1} \delta_{Q'}, C)\text{-basis at } (\mathbf{y}', M_0, \mathbf{p}^{\mathbf{y}'}).$$

From (44), (45), (48), we have

$$(51) \quad |\mathbf{y} - \mathbf{y}'| \leq C\delta_{Q'} = C\epsilon (\epsilon^{-1} \delta_{Q'}).$$

We recall from (48) and the hypotheses of the Main Lemma for  $\mathcal{A}$  that

$$(52) \quad \epsilon^{-1} \delta_{Q'} \leq \epsilon^{-1} \delta_{Q_0} \leq \delta_{\max},$$

and we recall from Section II.3 that

$$(53) \quad \mathcal{A} \text{ is monotonic.}$$

Thanks to (50)  $\cdots$  (53), Lemma 9 in Section I.4 (with  $\hat{\mathcal{A}} = \mathcal{A}$ ,  $\hat{\mathbf{p}}^0 = \mathbf{p}^0$ ) produces a polynomial  $\mathbf{p}' \in \mathcal{P}$  such that

$$(54) \quad \vec{\Gamma}_{l(\mathcal{A})-2} \text{ has an } (\mathcal{A}, \epsilon^{-1} \delta_{Q'}, C)\text{-basis at } (\mathbf{y}, M_0, \mathbf{p}');$$

$$(55) \quad \partial^\beta \left( \mathbf{p}' - \mathbf{p}^{\mathbf{y}'} \right) \equiv 0 \text{ for } \beta \in \mathcal{A};$$

and

$$(56) \quad \left| \partial^\beta \left( P' - P^{y'} \right) (y') \right| \leq CM_0 (\epsilon^{-1} \delta_{Q'})^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

From (54) we have in particular that

$$(57) \quad P' \in \Gamma_{l(\mathcal{A})-2}(y, CM_0),$$

and from (56) and (48) we obtain

$$(58) \quad \left| \partial^\beta \left( P^{y'} - P' \right) (y') \right| \leq CM_0 (\epsilon^{-1} \delta_Q)^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

If we knew that

$$(59) \quad \left| \partial^\beta (P^y - P') (y) \right| \leq M_0 (\epsilon^{-1} \delta_Q)^{m-|\beta|} \text{ for } \beta \in \mathcal{M},$$

then also  $\left| \partial^\beta (P^y - P') (y') \right| \leq C'M_0 (\epsilon^{-1} \delta_Q)^{m-|\beta|}$  for  $\beta \in \mathcal{M}$  since  $|y - y'| \leq C\delta_Q$  thanks to (44), (45), (48). Consequently, by (58), we would have  $\left| \partial^\beta \left( P^{y'} - P^y \right) (y') \right| \leq CM_0 (\epsilon^{-1} \delta_Q)^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ , which is our desired inequality (46). Thus, Lemma 14 will follow if we can prove (59).

Suppose (59) fails.

Corollary 1 shows that  $\vec{\Gamma}_{l(\mathcal{A})-1}$  has an  $(\mathcal{A}, \epsilon^{-1} \delta_Q, C)$ -basis at  $(y, M_0, P^y)$ . Since  $\Gamma_{l(\mathcal{A})-1}(x, M) \subset \Gamma_{l(\mathcal{A})-2}(x, M)$  for all  $x \in E$ ,  $M > 0$ , it follows that

$$(60) \quad \vec{\Gamma}_{l(\mathcal{A})-2} \text{ has an } (\mathcal{A}, \epsilon^{-1} \delta_Q, C)\text{-basis at } (y, M_0, P^y).$$

From (2) and (55) (applied to  $y$  and  $y'$ ), we see that

$$(61) \quad \partial^\beta (P^y - P') \equiv 0 \text{ for } \beta \in \mathcal{A}.$$

Since we are assuming that (59) fails, we have

$$(62) \quad \max_{\beta \in \mathcal{M}} (\epsilon^{-1} \delta_Q)^{|\beta|} \left| \partial^\beta (P^y - P') (y) \right| \geq M_0 (\epsilon^{-1} \delta_Q)^m.$$

Also, from (48) and the hypotheses of the Main Lemma for  $\mathcal{A}$ , we have

$$(63) \quad \epsilon^{-1} \delta_Q < \epsilon^{-1} \delta_{Q_0} \leq \delta_{\max}.$$

From Lemma 4 (B), we know that

$$(64) \quad \vec{\Gamma}_{l(\mathcal{A})-2} \text{ is } (C, \delta_{\max})\text{-convex}.$$

Our results (57), (60)  $\cdots$  (64) and Lemma 8 produce a set  $\hat{\mathcal{A}} \subseteq \mathcal{M}$  and a polynomial  $\hat{P} \in \mathcal{P}$ , with the following properties:

$$(65) \quad \hat{\mathcal{A}} \text{ is monotonic};$$

$$(66) \quad \hat{\mathcal{A}} < \mathcal{A} \text{ (strict inequality)};$$



$$(67) \quad \vec{\Gamma}_{l(\mathcal{A})-2} \text{ has an } (\hat{\mathcal{A}}, \epsilon^{-1}\delta_Q, C)\text{-basis at } (y, M_0, \hat{P});$$

$$(68) \quad \partial^\beta (\hat{P} - P^y) \equiv 0 \text{ for } \beta \in \mathcal{A} \text{ (recall, } \mathcal{A} \text{ is monotonic);}$$

and

$$(69) \quad |\partial^\beta (\hat{P} - P^y)(y)| \leq CM_0 (\epsilon^{-1}\delta_Q)^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

Now let  $z \in E \cap 5Q^+$ . We recall that  $\mathcal{A}$  is monotonic, and that (60), (61), (67), (68), (69) hold. Moreover, since  $y, z \in 5Q^+$ , we have  $|y - z| \leq C\delta_Q = C\epsilon(\epsilon^{-1}\delta_Q)$ . Thanks to the above remarks and the Small  $\epsilon$  Assumption, we may apply Lemma 9 to produce  $\check{P}^z \in \mathcal{P}$  satisfying the following conditions.

$$(70) \quad \vec{\Gamma}_{l(\mathcal{A})-3} \text{ has an } (\hat{\mathcal{A}}, \epsilon^{-1}\delta_Q, C)\text{-basis at } (z, M_0, \check{P}^z).$$

$$(71) \quad \partial^\beta (\check{P}^z - P^y) \equiv 0 \text{ for } \beta \in \mathcal{A}.$$

$$(72) \quad |\partial^\beta (\check{P}^z - P^y)(y)| \leq CM_0 (\epsilon^{-1}\delta_Q)^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

By (70) and the Large  $A$  Assumption,

$$(73) \quad \vec{\Gamma}_{l(\mathcal{A})-3} \text{ has an } (\hat{\mathcal{A}}, \epsilon^{-1}\delta_{Q^+}, A)\text{-basis at } (z, M_0, \check{P}^z).$$

By (2) and (71), we have

$$(74) \quad \partial^\beta (\check{P}^z - P^0) \equiv 0 \text{ for } \beta \in \mathcal{A}.$$

By (48) and (72), we have  $|\partial^\beta (\check{P}^z - P^y)(y)| \leq CM_0 (\epsilon^{-1}\delta_{Q_0})^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ , hence  $|\partial^\beta (\check{P}^z - P^y)(x_0)| \leq CM_0 (\epsilon^{-1}\delta_{Q_0})^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ , since  $x_0, y \in 5Q_0^+$ . Together with (3) and the Large  $A$  Assumption, this yields the estimate

$$(75) \quad |\partial^\beta (\check{P}^z - P^0)(x_0)| \leq AM_0 (\epsilon^{-1}\delta_{Q_0})^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

We have proven (73), (74), (75) for each  $z \in E \cap 5Q^+$ . Thus,  $5Q^+ \subset 5Q_0$  (see (49)),  $\hat{\mathcal{A}} < \mathcal{A}$  (strict inequality; see (66)), and for each  $z \in E \cap 5Q^+$  there exists  $\check{P}^z \in \mathcal{P}$  satisfying (73), (74), (75).

Comparing the above results with the definition of an OK cube, we see that  $Q^+$  is OK. On the other hand  $Q^+$  cannot be OK, since it properly contains the CZ cube  $Q$ . Assuming that (59) fails, we have derived a contradiction. Thus, (59) holds, completing the proof of Lemma 14. ■

## II.6 Good News About CZ Cubes

In this section we again place ourselves in the setting of Section II.3, and we make use of the auxiliary polynomials  $P^y$  and the CZ cubes  $Q$  defined above.

**Lemma 15** *Let  $Q \in CZ$ , with*

$$(1) \quad \frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset$$

*and*

$$(2) \quad \#(E \cap 5Q) \geq 2.$$

*Let*

$$(3) \quad y \in E \cap 5Q.$$

*Then there exist a set  $\mathcal{A}^\# \subseteq \mathcal{M}$  and a polynomial  $P^\# \in \mathcal{P}$  with the following properties.*

$$(4) \quad \mathcal{A}^\# \text{ is monotonic.}$$

$$(5) \quad \mathcal{A}^\# < \mathcal{A} \text{ (strict inequality).}$$

$$(6) \quad \vec{\Gamma}_{l(\mathcal{A})-3} \text{ has an } (\mathcal{A}^\#, \epsilon^{-1}\delta_Q, C(\mathcal{A}))\text{-basis at } (y, M_0, P^\#).$$

$$(7) \quad |\partial^\beta (P^\# - P^y)(y)| \leq C(\mathcal{A}) M_0 (\epsilon^{-1}\delta_Q)^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

**Proof.** Recall that

$$(8) \quad \partial^\beta (P^y - P^0) \equiv 0 \text{ for } \beta \in \mathcal{A} \text{ (see (2) in Section II.5)}$$

and that

$$(9) \quad 5Q \subseteq 5Q_0, \text{ since } Q \text{ is OK.}$$

Thanks to (3) and (9), Corollary 1 in Section II.5 applies, and it tells us that

$$(10) \quad \vec{\Gamma}_{l(\mathcal{A})-1} \text{ has an } (\mathcal{A}, \epsilon^{-1}\delta_Q, C)\text{-basis at } (y, M_0, P^y).$$

On the other hand,  $Q$  is OK and  $\#(E \cap 5Q) \geq 2$ ; hence, there exist  $\hat{\mathcal{A}} \subseteq \mathcal{M}$  and  $\hat{P} \in \mathcal{P}$  with the following properties

$$(11) \quad \vec{\Gamma}_{l(\mathcal{A})-3} \text{ has a weak } (\hat{\mathcal{A}}, \epsilon^{-1}\delta_Q, A)\text{-basis at } (y, M_0, \hat{P}).$$

$$(12) \quad |\partial^\beta (\hat{P} - P^0)(x_0)| \leq A M_0 (\epsilon^{-1}\delta_{Q_0})^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

$$(13) \quad \partial^\beta (\hat{P} - P^0) \equiv 0 \text{ for } \beta \in \mathcal{A}.$$

$$(14) \quad \hat{\mathcal{A}} < \mathcal{A} \text{ (strict inequality).}$$

We consider separately two cases.

Case 1: Suppose that

$$(15) \quad |\partial^\beta (\hat{\mathbf{P}} - \mathbf{P}^y)(\mathbf{y})| \leq M_0 (\epsilon^{-1} \delta_Q)^{m-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

By Lemma 4 (B),

$$(16) \quad \vec{\Gamma}_{\Gamma(\mathcal{A})-3} \text{ is } (C, \delta_{\max})\text{-convex.}$$

Also, (9) and hypothesis (A2) of the Main Lemma for  $\mathcal{A}$  give

$$(17) \quad \epsilon^{-1} \delta_Q \leq \epsilon^{-1} \delta_{Q_0} \leq \delta_{\max}.$$

Applying (11), (16), (17) and Lemma 7, we obtain a set  $\mathcal{A}^\# \subseteq \mathcal{M}$  such that

$$(18) \quad \mathcal{A}^\# \leq \hat{\mathcal{A}},$$

$$(19) \quad \mathcal{A}^\# \text{ is monotonic,}$$

and

$$(20) \quad \vec{\Gamma}_{\Gamma(\mathcal{A})-3} \text{ has an } (\mathcal{A}^\#, \epsilon^{-1} \delta_Q, C(\mathcal{A}))\text{-basis at } (\mathbf{y}, M_0, \hat{\mathbf{P}}).$$

Setting  $\mathbf{P}^\# = \hat{\mathbf{P}}$ , we obtain the desired conclusions (4)  $\cdots$  (7) at once from (14), (15), (18), (19), and (20).

Thus, Lemma 15 holds in Case 1.

Case 2: Suppose that  $|\partial^\beta (\hat{\mathbf{P}} - \mathbf{P}^y)(\mathbf{y})| > M_0 (\epsilon^{-1} \delta_Q)^{m-|\beta|}$  for some  $\beta \in \mathcal{M}$ , i.e.,

$$(21) \quad \max_{\beta \in \mathcal{M}} (\epsilon^{-1} \delta_Q)^{|\beta|} |\partial^\beta (\hat{\mathbf{P}} - \mathbf{P}^y)(\mathbf{y})| > M_0 (\epsilon^{-1} \delta_Q)^m.$$

From (11) we have

$$(22) \quad \hat{\mathbf{P}} \in \Gamma_{\Gamma(\mathcal{A})-3}(\mathbf{y}, \lambda M_0).$$

Since  $\Gamma_{\Gamma(\mathcal{A})-1}(\mathbf{x}, M) \subseteq \Gamma_{\Gamma(\mathcal{A})-3}(\mathbf{x}, M)$  for all  $\mathbf{x} \in E, M > 0$ , (10) implies that

$$(23) \quad \vec{\Gamma}_{\Gamma(\mathcal{A})-3} \text{ has an } (\mathcal{A}, \epsilon^{-1} \delta_Q, C)\text{-basis at } (\mathbf{y}, M_0, \mathbf{P}^y).$$

As in Case 1,

$$(24) \quad \vec{\Gamma}_{\Gamma(\mathcal{A})-3} \text{ is } (C, \delta_{\max})\text{-convex,}$$

and

$$(25) \quad \epsilon^{-1} \delta_Q \leq \epsilon^{-1} \delta_{Q_0} \leq \delta_{\max}.$$

From (8) and (13) we have

$$(26) \quad \partial^\beta (\hat{\mathbf{P}} - \mathbf{P}^y) \equiv 0 \text{ for } \beta \in \mathcal{A}.$$

Thanks to (21)  $\cdots$  (26) and Lemma 8 there exist  $\mathcal{A}^\# \subseteq \mathcal{M}$  and  $P^\# \in \mathcal{P}$  with the following properties:  $\mathcal{A}^\#$  is monotonic;  $\mathcal{A}^\# < \mathcal{A}$  (strict inequality);  $\vec{\Gamma}_{\mathfrak{l}(\mathcal{A})-3}$  has an  $(\mathcal{A}^\#, \epsilon^{-1}\delta_Q, C(\mathcal{A}))$ -basis at  $(y, M_0, P^\#)$ ;  $\partial^\beta(P^\# - P^y) \equiv 0$  for  $\beta \in \mathcal{A}$ ;  $|\partial^\beta(P^\# - P^y)(y)| \leq M_0(\epsilon^{-1}\delta_Q)^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ .

Thus,  $\mathcal{A}^\#$  and  $P^\#$  satisfy (4)  $\cdots$  (7), proving Lemma 15 in Case 2.

We have seen that Lemma 15 holds in all cases. ■

**Remarks** • *The analysis of Case 2 in the proof of Lemma 15 is a new ingredient, with no analogue in our previous work on Whitney problems.*

- *The proof of Lemma 15 gives a  $\hat{P}$  that satisfies also  $\partial^\beta(\hat{P} - P^0) \equiv 0$  for  $\beta \in \mathcal{A}$ , but we make no use of that.*
- *Note that  $x_0$  and  $\delta_{Q_0}$  appear in (12), rather than the desired  $y, \delta_Q$ . Consequently, (12) is of no help in the proof of Lemma 15.*

In the proof of our next result, we use our Induction Hypothesis that the Main Lemma for  $\mathcal{A}'$  holds whenever  $\mathcal{A}' < \mathcal{A}$  and  $\mathcal{A}'$  is monotonic. (See Section II.3.)

**Lemma 16** *Let  $Q \in CZ$ . Suppose that*

$$(27) \quad \frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset$$

*and*

$$(28) \quad \#(E \cap 5Q) \geq 2.$$

*Let*

$$(29) \quad y \in E \cap 5Q.$$

*Then there exists  $F^{y,Q} \in C^m(\frac{65}{64}Q)$  such that*

$$(*1) \quad |\partial^\beta(F^{y,Q} - P^y)| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|} \text{ on } \frac{65}{64}Q, \text{ for } |\beta| \leq m; \text{ and}$$

$$(*2) \quad J_z(F^{y,Q}) \in \Gamma_0(z, C(\epsilon) M_0) \text{ for all } z \in E \cap \frac{65}{64}Q.$$

**Proof.** Our hypotheses (27), (28), (29) are precisely the hypotheses of Lemma 15. Let  $\mathcal{A}^\#, P^\#$  satisfy the conclusions (4)  $\cdots$  (7) of that Lemma. Recall the definition of  $\mathfrak{l}(\mathcal{A})$ ; see (1), (2) in Section II.1. We have  $\mathfrak{l}(\mathcal{A}^\#) \leq \mathfrak{l}(\mathcal{A}) - 3$  since  $\mathcal{A}^\# < \mathcal{A}$ ; hence (6) implies that

$$(30) \quad \vec{\Gamma}_{\mathfrak{l}(\mathcal{A}^\#)} \text{ has an } (\mathcal{A}^\#, \epsilon^{-1}\delta_Q, C(\mathcal{A}))\text{-basis at } (y, M_0, P^\#).$$

Also, since  $Q$  is OK, we have  $5Q \subseteq 5Q_0$ , hence  $\delta_Q \leq \delta_{Q_0}$ . Hence, hypothesis (A2) of the Main Lemma for  $\mathcal{A}$  implies that

$$(31) \quad \epsilon^{-1}\delta_Q \leq \delta_{\max}.$$

By (4), (5), and our Inductive Hypothesis, the Main Lemma holds for  $\mathcal{A}^\#$ . Thanks to (29), (30), (31) and the Small  $\epsilon$  Assumption in Section II.3, the Main Lemma for  $\mathcal{A}^\#$  now yields a function  $F \in C^m(\frac{65}{64}Q)$ , such that

$$(32) \quad |\partial^\beta (F - P^\#)| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|} \text{ on } \frac{65}{64}Q, \text{ for } |\beta| \leq m; \text{ and}$$

$$(33) \quad J_z(F) \in \Gamma_0(z, C(\epsilon) M_0) \text{ for all } z \in E \cap \frac{65}{64}Q.$$

Thanks to conclusion (7) of Lemma 15 (together with (29)), we have also

$$(34) \quad |\partial^\beta (P^\# - P^\mathfrak{y})| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|} \text{ on } \frac{65}{64}Q \text{ for } |\beta| \leq m.$$

(Recall that  $P^\# - P^\mathfrak{y}$  is a polynomial of degree at most  $m - 1$ .) Taking  $F^\mathfrak{y}, Q = F$ , we may read off the desired conclusions (\*1) and (\*2) from (32), (33), (34).

The proof of Lemma 16 is complete. ■

## II.7 Local Interpolants

In this section, we again place ourselves in the setting of Section II.3. We make use of the Calderón-Zygmund cubes  $Q$  and the auxiliary polynomials  $P^\mathfrak{y}$  defined above. Let

$$(1) \quad \mathcal{Q} = \{Q \in \mathcal{CZ} : \frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset\}.$$

For each  $Q \in \mathcal{Q}$ , we define a function  $F^Q \in C^m(\frac{65}{64}Q)$  and a polynomial  $P^Q \in \mathcal{P}$ . We proceed by cases. We say that  $Q \in \mathcal{Q}$  is

**Type 1** if  $\#(E \cap 5Q) \geq 2$ ,

**Type 2** if  $\#(E \cap 5Q) = 1$ ,

**Type 3** if  $\#(E \cap 5Q) = 0$  and  $\delta_Q \leq \frac{1}{1024}\delta_{Q_0}$ , and

**Type 4** if  $\#(E \cap 5Q) = 0$  and  $\delta_Q > \frac{1}{1024}\delta_{Q_0}$ .

If  $Q$  is of Type 1, then we pick a point  $y_Q \in E \cap 5Q$ , and set  $P^Q = P^{y_Q}$ . Applying Lemma 16, we obtain a function  $F^Q \in C^m(\frac{65}{64}Q)$  such that

$$(2) \quad |\partial^\beta (F^Q - P^Q)| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|} \text{ on } \frac{65}{64}Q, \text{ for } |\beta| \leq m; \text{ and}$$

$$(3) \quad J_z(F^Q) \in \Gamma_0(z, C(\epsilon) M_0) \text{ for all } z \in E \cap \frac{65}{64}Q.$$

If  $Q$  is of Type 2, then we let  $y_Q$  be the one and only point of  $E \cap 5Q$ , and define  $F^Q = P^Q = P^{y_Q}$ . Then (2) holds trivially. If  $y_Q \notin \frac{65}{64}Q$  then (3) holds vacuously.

If  $y_Q \in \frac{65}{64}Q$ , then (3) asserts that  $P^{y_Q} \in \Gamma_0(y_Q, C(\epsilon)M_0)$ . Thanks to (2) in Section II.5, we know that  $P^{y_Q} \in \Gamma_{l(\mathcal{A})-1}(y_Q, CM_0) \subset \Gamma_0(y_Q, C(\epsilon)M_0)$ . Thus, (2) and (3) hold also when  $Q$  is of Type 2.

If  $Q$  is of Type 3, then  $5Q^+ \subset 5Q_0$ , since  $\frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset$  and  $\delta_Q \leq \frac{1}{1024}\delta_{Q_0}$ . However,  $Q^+$  cannot be OK, since  $Q$  is a CZ cube. Therefore  $\#(E \cap 5Q^+) \geq 2$ . We pick  $y_Q \in E \cap 5Q^+$ , and set  $F^Q = P^Q = P^{y_Q}$ . Then (2) holds trivially, and (3) holds vacuously.

If  $Q$  is of Type 4, then we set  $F^Q = P^Q = P^0$ , and again (2) holds trivially, and (3) holds vacuously.

Note that if  $Q$  is of Type 1, 2, or 3, then we have defined a point  $y_Q$ , and we have  $P^Q = P^{y_Q}$  and

$$(4) \quad y_Q \in E \cap 5Q^+ \cap 5Q_0.$$

(If  $Q$  is of Type 1 or 2, then  $y_Q \in E \cap 5Q$  and  $5Q \subseteq 5Q_0$  since  $Q$  is OK. If  $Q$  is of Type 3, then  $y_Q \in E \cap 5Q^+$  and  $5Q^+ \subset 5Q_0$ ). We have picked  $F^Q$  and  $P^Q$  for all  $Q \in \mathcal{Q}$ , and (2), (3) hold in all cases.

**Lemma 17 (“Consistency of the  $P^Q$ ”)** *Let  $Q, Q' \in \mathcal{Q}$ , and suppose  $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$ . Then*

$$(5) \quad \left| \partial^\beta (P^Q - P^{Q'}) \right| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|} \text{ on } \frac{65}{64}Q \cap \frac{65}{64}Q', \text{ for } |\beta| \leq m.$$

**Proof.** Suppose first that neither  $Q$  nor  $Q'$  is Type 4. Then  $P^Q = P^{y_Q}$  and  $P^{Q'} = P^{y_{Q'}}$  with  $y_Q \in E \cap 5Q^+ \cap 5Q_0$ ,  $y_{Q'} \in E \cap 5(Q')^+ \cap 5Q_0$ . Thanks to Lemma 14, we have

$$\left| \partial^\beta (P^Q - P^{Q'}) (y_Q) \right| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|} \text{ for } \beta \in \mathcal{M},$$

which implies (5), since  $y_Q \in 5Q^+$  and  $P^Q - P^{Q'}$  is an  $(m-1)^{\text{rst}}$  degree polynomial.

Next, suppose that  $Q$  and  $Q'$  are both Type 4.

Then by definition  $P^Q = P^{Q'} = P^0$ , and consequently (5) holds trivially.

Finally, suppose that exactly one of  $Q, Q'$  is of Type 4.

Since  $\delta_Q$  and  $\delta_{Q'}$ , differ by at most a factor of 2, the cubes  $Q$  and  $Q'$  may be interchanged without loss of generality. Hence, we may assume that  $Q'$  is of Type 4 and  $Q$  is not. By definition of Type 4,

$$(6) \quad \delta_{Q'} > \frac{1}{1024}\delta_{Q_0}; \text{ hence also } \delta_Q \geq \frac{1}{1024}\delta_{Q_0},$$

since  $\delta_Q, \delta_{Q'}$ , are powers of 2 that differ by at most a factor of 2.

Since  $Q'$  is of Type 4 and  $Q$  is not, we have  $P^Q = P^{y_Q}$  and  $P^{Q'} = P^0$ , with

$$(7) \quad y_Q \in E \cap 5Q^+ \cap 5Q_0.$$

Thus, in this case, (5) asserts that

$$(8) \quad |\partial^\beta (P^{y_Q} - P^0)| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|} \text{ on } \frac{65}{64}Q \cap \frac{65}{64}Q', \text{ for } |\beta| \leq m.$$

However, by (7) above, property (3) in Section II.5 gives the estimate

$$(9) \quad |\partial^\beta (P^{y_Q} - P^0)(x_0)| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Recall from the hypotheses of the Main Lemma for  $\mathcal{A}$  that  $x_0 \in 5(Q_0)^+$ . Since  $P^{y_Q} - P^0$  is an  $(m-1)^{\text{rst}}$  degree polynomial, we conclude from (9) that

$$(10) \quad |\partial^\beta (P^{y_Q} - P^0)| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|} \text{ on } 5Q, \text{ for } |\beta| \leq m.$$

The desired inequality (8) now follows from (6) and (10). Thus, (5) holds in all cases.

The proof of Lemma 17 is complete. ■

From estimate (2), Lemma 11 and Lemma 17, we immediately obtain the following.

**Corollary 2** *Let  $Q, Q' \in \mathcal{Q}$  and suppose that  $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$ . Then*

$$(11) \quad |\partial^\beta (F^Q - F^{Q'})| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|} \text{ on } \frac{65}{64}Q \cap \frac{65}{64}Q', \text{ for } |\beta| \leq m.$$

Regarding the polynomials  $P^Q$ , we make the following simple observation.

**Lemma 18** *We have*

$$(12) \quad |\partial^\beta (P^Q - P^0)| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|} \text{ on } \frac{65}{64}Q, \text{ for } |\beta| \leq m \text{ and } Q \in \mathcal{Q}.$$

**Proof.** Recall that if  $Q$  is of Type 1, 2, or 3, then  $P^Q = P^{y_Q}$  for some  $y_Q \in E \cap 5Q_0$ . From estimate (3) in Section II.5, we know that

$$(13) \quad |\partial^\beta (P^Q - P^0)(x_0)| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Since  $x_0 \in 5Q_0^+$  (see the hypotheses of the Main Lemma for  $\mathcal{A}$ ) and  $P^Q - P^0$  is a polynomial of degree at most  $m-1$ , and since  $\frac{65}{64}Q \subset 5Q \subset 5Q_0$  (because  $Q$  is OK), estimate (13) implies the desired estimate (12).

If instead,  $Q$  is of Type 4, then by definition  $P^Q = P^0$ , hence estimate (12) holds trivially.

Thus, (12) holds in all cases. ■

**Corollary 3** *For  $Q \in \mathcal{Q}$  and  $|\beta| \leq m$ , we have  $|\partial^\beta (F^Q - P^0)| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|}$  on  $\frac{65}{64}Q$ .*

**Proof.** Recall that, since  $Q$  is OK, we have  $5Q \subset 5Q_0$ . The desired estimate therefore follows from estimates (2) and (12). ■

## II.8 Completing the Induction

We again place ourselves in the setting of Section II.3. We use the CZ cubes  $Q$  and the functions  $F^Q$  defined above. We recall several basic results from earlier sections.

- (1)  $\vec{\Gamma}_0$  is a  $(C, \delta_{\max})$ -convex shape field.
- (2)  $\epsilon^{-1} \delta_{Q_0} \leq \delta_{\max}$ , hence  $\epsilon^{-1} \delta_Q \leq \delta_{\max}$  for  $Q \in CZ$ .
- (3) The cubes  $Q \in CZ$  partition the interior of  $5Q_0$ .
- (4) For  $Q, Q' \in CZ$ , if  $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$ , then  $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$ .

Let

- (5)  $\mathcal{Q} = \{Q \in CZ : \frac{65}{64}Q \cap \frac{65}{64}Q_0 \neq \emptyset\}$ .

Then

- (6)  $\mathcal{Q}$  is finite.

For each  $Q \in \mathcal{Q}$ , we have

- (7)  $F^Q \in C^m(\frac{65}{64}Q)$ ,
- (8)  $J_z(F^Q) \in \Gamma_0(z, C(\epsilon)M_0)$  for  $z \in E \cap \frac{65}{64}Q$ , and
- (9)  $|\partial^\beta(F^Q - P^0)| \leq C(\epsilon)M_0\delta_{Q_0}^{m-|\beta|}$  on  $\frac{65}{64}Q$ , for  $|\beta| \leq m$ .
- (10) For each  $Q, Q' \in \mathcal{Q}$ , if  $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$ , then  $|\partial^\beta(F^Q - F^{Q'})| \leq C(\epsilon)M_0\delta_Q^{m-|\beta|}$  on  $\frac{65}{64}Q \cap \frac{65}{64}Q'$ , for  $|\beta| \leq m$ .

We introduce a Whitney partition of unity adapted to the cubes  $Q \in CZ$ . For each  $Q \in CZ$ , let  $\tilde{\theta}_Q \in C^m(\mathbb{R}^n)$  satisfy  $\tilde{\theta}_Q = 1$  on  $Q$ ,  $\text{support}(\tilde{\theta}_Q) \subset \frac{65}{64}Q$ ,

$$|\partial^\beta \tilde{\theta}_Q| \leq C\delta_Q^{-|\beta|} \text{ for } |\beta| \leq m.$$

Set

$$(11) \quad \theta_Q = \tilde{\theta}_Q \cdot \left( \sum_{Q' \in CZ} (\tilde{\theta}_{Q'})^2 \right)^{-1/2}.$$

Note that we have

$$(12) \quad \sum_{Q \in \mathcal{Q}} \theta_Q^2 = 1 \text{ on } \frac{65}{64}Q_0.$$

We define

$$(13) \quad F = \sum_{Q \in \mathcal{Q}} \theta_Q^2 F^Q.$$



For each  $Q \in \mathcal{Q}$ , (7), (11) show that  $\theta_Q^2 F^Q \in C^m(\mathbb{R}^n)$ . Since also  $\mathcal{Q}$  is finite (see (6)), it follows that

$$(14) \quad F \in C^m(\mathbb{R}^n).$$

Moreover, for any  $x \in \frac{65}{64}Q_0$  and any  $\beta$  of order  $|\beta| \leq m$ , we have

$$(15) \quad \partial^\beta F(x) = \sum_{Q \in \mathcal{Q}(x)} \partial^\beta \left\{ \theta_Q^2 F^Q \right\}, \text{ where}$$

$$(16) \quad \mathcal{Q}(x) = \{Q \in \mathcal{Q} : x \in \frac{65}{64}Q\}.$$

Note that

$$(17) \quad \#(\mathcal{Q}(x)) \leq C, \text{ by (4).}$$

Let  $\hat{Q}$  be the CZ cube containing  $x$ . (There is one and only one such cube, thanks to (3); recall that we suppose that  $x \in \frac{65}{64}Q_0$ .) Then  $\hat{Q} \in \mathcal{Q}(x)$ , and (15) may be written in the form

$$(18) \quad \partial^\beta (F - P^0)(x) = \partial^\beta (F^{\hat{Q}} - P^0)(x) + \sum_{Q \in \mathcal{Q}(x)} \partial^\beta \left\{ \theta_Q^2 \cdot (F^Q - F^{\hat{Q}}) \right\}(x).$$

(Here we use (12).) The first term on the right in (18) has absolute value at most  $C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|}$ ; see (9). At most  $C$  distinct cubes  $Q$  enter into the second term on the right in (18); see (17). For each  $Q \in \mathcal{Q}(x)$ , we have

$$\left| \partial^\beta \left\{ \theta_Q^2 \cdot (F^Q - F^{\hat{Q}}) \right\}(x) \right| \leq C(\epsilon) M_0 \delta_Q^{m-|\beta|},$$

by (10) and (11). Hence, for each  $Q \in \mathcal{Q}(x)$ , we have

$$\left| \partial^\beta \left\{ \theta_Q^2 \cdot (F^Q - F^{\hat{Q}}) \right\}(x) \right| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|},$$

see (3).

The above remarks and (18) together yield the estimate

$$(19) \quad \left| \partial^\beta (F - P^0) \right| \leq C(\epsilon) M_0 \delta_{Q_0}^{m-|\beta|} \text{ on } \frac{65}{64}Q_0, \text{ for } |\beta| \leq m.$$

Moreover, let  $z \in E \cap \frac{65}{64}Q_0$ . Then

$$J_z(F) = \sum_{Q \in \mathcal{Q}(z)} J_z(\theta_Q) \odot_z J_z(\theta_Q) \odot_z J_z(F^Q) \quad (\text{see (15)});$$

$$\left| \partial^\beta [J_z(\theta_Q)](z) \right| \leq C \delta_Q^{|\beta|} \text{ for } |\beta| \leq m-1, Q \in \mathcal{Q}(z) \quad (\text{see (11)});$$

$$\sum_{Q \in \mathcal{Q}(z)} [J_z(\theta_Q)] \odot_z [J_z(\theta_Q)] = 1$$

(see (12) and note that  $J_z(\theta_Q) = 0$  for  $Q \notin \mathcal{Q}(z)$  by (11) and (16));

$$J_z(F^Q) \in \Gamma_0(z, C(\epsilon)M_0) \text{ for } Q \in \mathcal{Q}(z) \text{ (see (8))};$$

$$\left| \partial^\beta \left\{ J_z(F^Q) - J_z(F^{Q'}) \right\} (z) \right| \leq C(\epsilon)M_0\delta_Q^{m-|\beta|}$$

for  $|\beta| \leq m-1$ ,  $Q, Q' \in \mathcal{Q}(z)$  (see (10));  $\#(\mathcal{Q}(z)) \leq C$  (see (17));  $\delta_Q \leq \delta_{\max}$  (see (2));  $\vec{\Gamma}_0$  is a  $(C, \delta_{\max})$ -convex shape field (see (1)).

The above results, together with Lemma 2, tell us that

$$(20) \quad J_z(F) \in \Gamma_0(z, C(\epsilon)M_0) \text{ for all } z \in E \cap \frac{65}{64}Q_0.$$

From (14), (19), (20), we see at once that the restriction of  $F$  to  $\frac{65}{64}Q_0$  belongs to  $C^m(\frac{65}{64}Q_0)$  and satisfies conditions (C\*1) and (C\*2) in Section II.3. As we explained in that section, once we have found a function in  $C^m(\frac{65}{64}Q_0)$  satisfying (C\*1) and (C\*2), our induction on  $\mathcal{A}$  is complete. Thus, we have proven the Main Lemma for all monotonic  $\mathcal{A} \subseteq \mathcal{M}$ . ■

## II.9 Restatement of the Main Lemma

An equivalent version of the Main Lemma for  $\mathcal{A} = \emptyset$  reads as follows.

**Restated Main Lemma** *Let  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field. For  $l \geq 1$ , let  $\vec{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0}$  be the  $l^{\text{th}}$ -refinement of  $\vec{\Gamma}_0$ . Fix a dyadic cube  $Q_0$  of sidelength  $\delta_{Q_0} \leq \epsilon\delta_{\max}$ , where  $\epsilon > 0$  is a small enough constant determined by  $m, n, C_w$ . Let  $x_0 \in E \cap 5Q_0^+$ , and let  $P_0 \in \Gamma_{l(\emptyset)}(x_0, M_0)$ .*

*Then there exists a function  $F \in C^m(\frac{65}{64}Q_0)$ , satisfying*

- $|\partial^\beta(F - P_0)(x)| \leq C_*M_0\delta_{Q_0}^{m-|\beta|}$  for  $x \in \frac{65}{64}Q_0$ ,  $|\beta| \leq m$ ; and
- $J_z(F) \in \Gamma_0(z, C_*M_0)$  for all  $z \in E \cap \frac{65}{64}Q_0$ ;

where  $C_*$  is determined by  $C_w, m, n$ .

## II.10 Tidying Up

In this section, we remove from the Restated Main Lemma the small constant  $\epsilon$  and the assumption that  $Q_0$  is dyadic.

**Theorem 5** *Let  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field. For  $l \geq 1$ , let  $\vec{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0}$  be the  $l^{\text{th}}$ -refinement of  $\vec{\Gamma}_0$ . Fix a cube  $Q_0$  of sidelength  $\delta_{Q_0} \leq \delta_{\max}$ , a point  $x_0 \in E \cap 5Q_0$ , and a real number  $M_0 > 0$ . Let  $P_0 \in \Gamma_{l(\emptyset)+1}(x_0, M_0)$ .*

*Then there exists a function  $F \in C^m(Q_0)$  satisfying the following, with  $C_*$  determined by  $C_w, m, n$ .*

- $|\partial^\beta (F - P_0)(x)| \leq C_* M_0 \delta_{Q_0}^{m-|\beta|}$  for  $x \in Q_0$ ,  $|\beta| \leq m$ ; and
- $J_z(F) \in \Gamma_0(z, C_* M_0)$  for all  $z \in E \cap Q_0$ .

**Sketch of Proof.** Let  $\epsilon > 0$  be the small constant in the statement of the Restated Main Lemma in Section II.9. In particular,  $\epsilon$  is determined by  $C_w$ ,  $m$ ,  $n$ . We write  $c$ ,  $C$ ,  $C'$ , etc., to denote constants determined by  $C_w$ ,  $m$ ,  $n$ . These symbols may denote different constants in different occurrences.

We cover  $CQ_0$  by a grid of dyadic cubes  $\{Q_v\}$ , all having the same sidelength  $\delta_{Q_v}$ , with  $\frac{\epsilon}{20}\delta_{Q_0} \leq \delta_{Q_v} \leq \epsilon\delta_{Q_0}$ , and all contained in  $C'Q_0$ .

For each  $Q_v$  with  $E \cap \frac{65}{64}Q_v \neq \emptyset$ , we pick a point  $x_v \in E \cap \frac{65}{64}Q_v$ ; by definition of the  $l^{\text{th}}$ -refinement, there exists  $P_v \in \Gamma_{l(\emptyset)}(x_v, M_0)$  such that  $|\partial^\beta (P_v - P_0)(x_0)| \leq CM_0 \delta_{Q_0}^{m-|\beta|}$  for  $\beta \in \mathcal{M}$ .

Since  $x_v \in E \cap \frac{65}{64}Q_v$ ,  $P_v \in \Gamma_{l(\emptyset)}$ , and  $\delta_{Q_v} \leq \epsilon\delta_{Q_0} \leq \epsilon\delta_{\max}$ , the Restated Main Lemma applies to  $x_v, P_v, Q_v$  to produce  $F_v \in C^m(\frac{65}{64}Q_v)$  satisfying

$$(1) \quad |\partial^\beta (F_v - P_v)(x)| \leq CM_0 \delta_{Q_v}^{m-|\beta|} \leq CM_0 \delta_{Q_0}^{m-|\beta|} \text{ for } x \in \frac{65}{64}Q_v, |\beta| \leq m;$$

and

$$(2) \quad J_z(F_v) \in \Gamma_0(z, CM_0) \text{ for all } z \in E \cap \frac{65}{64}Q_v.$$

We have produced such  $F_v$  for those  $v$  satisfying  $E \cap \frac{65}{64}Q_v \neq \emptyset$ . If instead  $E \cap \frac{65}{64}Q_v = \emptyset$ , then we set  $F_v = P_0$ .

Next, we introduce a partition of unity. We fix cutoff functions  $\theta_v \in C^m(\mathbb{R}^n)$  satisfying

$$(3) \quad \text{support } \theta_v \subset \frac{65}{64}Q_v, \quad |\partial^\beta \theta_v| \leq C \delta_{Q_0}^{-|\beta|} \text{ for } |\beta| \leq m, \quad \sum_v \theta_v^2 = 1 \text{ on } Q_0.$$

We then define

$$(4) \quad F = \sum_v \theta_v^2 F_v \text{ on } Q_0.$$

One checks easily that  $F$  satisfies the conclusions of Theorem 5. ■

## Part III

# Applications

## III.1 Finiteness Principle I

In this section we prove a finiteness principle for shape fields.

Let  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  be a shape field. For  $l \geq 1$ , let  $\vec{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0}$  be the  $l^{\text{th}}$ -refinement of  $\vec{\Gamma}_0$ . Fix  $M_0 > 0$ . For  $x \in E$ ,  $S \subset E$ , define

$$(1) \Gamma(x, S) = \left\{ \begin{array}{l} P^x : \vec{P} = (P^y)_{y \in S \cup \{x\}} \in \text{Wh}(S \cup \{x\}), \|\vec{P}\|_{\dot{C}^m(S \cup \{x\})} \leq M_0, \\ P^y \in \Gamma_0(y, M_0) \text{ for all } y \in S \cup \{x\}. \end{array} \right\}$$

(See Section I.1 for the definition of  $\text{Wh}(\cdot)$  and  $\|\cdot\|_{\dot{C}^m(\cdot)}$ .) Note that

$$(2) \Gamma(x, \emptyset) = \Gamma_0(x, M_0).$$

Define

$$(3) \Gamma_l^{\text{fp}}(x, M_0) = \bigcap_{S \subset E, \#(S) \leq (D+2)^l} \Gamma(x, S) \text{ for } l \geq 0, \text{ where}$$

$$(4) D = \dim \mathcal{P}.$$

Note that

$$(5) \Gamma_0^{\text{fp}}(x, M_0) \subseteq \Gamma_0(x, M_0), \text{ thanks to (2).}$$

Each  $\Gamma(x, S)$ ,  $\Gamma_l^{\text{fp}}(x, M_0)$  is a (possibly empty) convex subset of  $\mathcal{P}$ .

As a consequence of Helly's theorem, we have the following result.

**Lemma 19** *Let  $x \in E$ ,  $l \geq 0$ . Suppose  $\Gamma(x, S) \neq \emptyset$  for all  $S \subset E$  with  $\#(S) \leq (D+2)^{l+1}$ . Then  $\Gamma_l^{\text{fp}}(x, M_0) \neq \emptyset$ .*

**Lemma 20** *For  $x \in E$ ,  $l \geq 0$ , we have*

$$(6) \Gamma_l^{\text{fp}}(x, M_0) \subseteq \Gamma_l(x, M_0).$$

**Proof.** We use induction on  $l$ . The base case  $l = 0$  is our observation (5). For the induction step, fix  $l \geq 1$ . We will prove (6) under the inductive assumption

$$(7) \Gamma_{l-1}^{\text{fp}}(y, M_0) \subseteq \Gamma_{l-1}(y, M_0) \text{ for all } y \in E.$$

Thus, let  $P \in \Gamma_l^{\text{fp}}(x, M_0)$  be given. We must prove that  $P \in \Gamma_l(x, M_0)$ , which means that given  $y \in E$  there exists

$$(8) P' \in \Gamma_{l-1}(y, M_0) \text{ such that } |\partial^\beta (P - P')(x)| \leq M_0 |x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

We will prove that there exists

$$(9) P' \in \Gamma_{l-1}^{\text{fp}}(y, M_0) \text{ such that } |\partial^\beta (P - P')(x)| \leq M_0 |x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Thanks to our inductive hypothesis (7), we see that (9) implies (8). Therefore, to complete the proof of the Lemma, it is enough to prove the existence of a  $P'$  satisfying (9). For  $S \subset E$ , define

$$\hat{\Gamma}(S) = \left\{ \begin{array}{l} P^y : \vec{P} = (P^z)_{z \in S \cup \{x, y\}} \in \text{Wh}(S \cup \{x, y\}), P^x = P, \|\vec{P}\|_{\dot{C}^m(S \cup \{x, y\})} \leq M_0, \\ P^z \in \Gamma_0(z, M_0) \text{ for all } z \in S \cup \{x, y\}. \end{array} \right\}$$

By definition,

$$(10) \quad \hat{\Gamma}(S) \subset \Gamma(y, S) \text{ for } S \subset E.$$

Let  $S_1, \dots, S_{D+1} \subset E$  with  $\#(S_i) \leq (D+2)^{l-1}$  for each  $i$ .

Then  $\hat{\Gamma}(S_1) \cap \dots \cap \hat{\Gamma}(S_{D+1}) \supset \hat{\Gamma}(S_1 \cup \dots \cup S_{D+1})$ , and  $\#(S_1 \cup \dots \cup S_{D+1} \cup \{y\}) \leq (D+1)(D+2)^{l-1} + 1 \leq (D+2)^l$ . Since  $P \in \Gamma_l^{\text{fp}}(x, M_0)$ , it follows that there exists  $\vec{P} = (P^z)_{z \in S_1 \cup \dots \cup S_{D+1} \cup \{x, y\}} \in \text{Wh}(S_1 \cup \dots \cup S_{D+1} \cup \{x, y\})$  such that  $P^x = P$ ,  $\|\vec{P}\|_{\dot{C}^m(S_1 \cup \dots \cup S_{D+1} \cup \{x, y\})} \leq M_0$ ,  $P^z \in \Gamma_0(z, M_0)$  for all  $z \in S_1 \cup \dots \cup S_{D+1} \cup \{x, y\}$ . We then have  $P^y \in \hat{\Gamma}(S_1 \cup \dots \cup S_{D+1})$ , hence  $\hat{\Gamma}(S_1) \cap \dots \cap \hat{\Gamma}(S_{D+1}) \supset \hat{\Gamma}(S_1 \cup \dots \cup S_{D+1}) \neq \emptyset$ .

By Helly's theorem, there exists

$$(11) \quad P' \in \bigcap_{S \subset E, \#(S) \leq (D+2)^{l-1}} \hat{\Gamma}(S).$$

In particular,  $P' \in \hat{\Gamma}(\emptyset)$ , which implies that

$$|\partial^\beta (P - P')(x)| \leq M_0 |x - y|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Also, (10), (11) imply that

$$P' \in \bigcap_{S \subset E, \#(S) \leq (D+2)^{l-1}} \Gamma(y, S) = \Gamma_{l-1}^{\text{fp}}(y, M_0).$$

Thus,  $P'$  satisfies (9), completing the proof of Lemma 20. ■

**Theorem 6 (Finiteness Principle for Shape Fields)** *For a large enough  $k^\#$  determined by  $m, n$ , the following holds. Let  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field and let  $Q_0 \subset \mathbb{R}^n$  be a cube of sidelength  $\delta_{Q_0} \leq \delta_{\max}$ . Also, let  $x_0 \in E \cap 5Q_0$  and  $M_0 > 0$  be given. Assume that for each  $S \subset E$  with  $\#(S) \leq k^\#$  there exists a Whitney field  $\vec{P}^S = (P^z)_{z \in S}$  such that*

$$\|\vec{P}^S\|_{\dot{C}^m(S)} \leq M_0,$$

and

$$P^z \in \Gamma_0(z, M_0) \text{ for all } z \in S.$$

Then there exist  $P^0 \in \Gamma_0(x_0, M_0)$  and  $F \in C^m(Q_0)$  such that the following hold, with a constant  $C_*$  determined by  $C_w, m, n$ :

- $J_z(F) \in \Gamma_0(z, C_* M_0)$  for all  $z \in E \cap Q_0$ .
- $|\partial^\beta (F - P^0)(x)| \leq C_* M_0 \delta_{Q_0}^{m-|\beta|}$  for all  $x \in Q_0$ ,  $|\beta| \leq m$ .
- In particular,  $|\partial^\beta F(x)| \leq C_* M_0$  for all  $x \in Q_0$ ,  $|\beta| = m$ .

**Proof.** For  $l \geq 1$ , define  $\vec{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0}$  and  $\vec{\Gamma}_l^{\text{fp}} = \left( \Gamma_l^{\text{fp}}(x, M) \right)_{x \in E, M > 0}$  as in Lemmas 19 and 20. We take  $l_* = 100 + l(\emptyset)$  and  $k^\# = 100 + (D + 2)^{l_* + 100}$ . (For the definition of  $l(\emptyset)$ , see Section II.1.)

Lemmas 19 and 20 show that  $\Gamma_{l_*}^{\text{fp}}(x_0, M_0)$  is nonempty, hence  $\Gamma_{l(\emptyset)+1}(x_0, M_0)$  is nonempty. Pick any  $P^0 \in \Gamma_{l(\emptyset)+1}(x_0, M_0) \subset \Gamma_0(x_0, M_0)$ . Then Theorem 5 in Section II.10 produces a function  $F \in C^m(Q_0)$  with the desired properties. ■

The finiteness principle for  $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$  stated in the Introduction follows easily from Theorem 6, under the following assumptions on  $\vec{\Gamma}$ :

- $\vec{\Gamma}$  is a  $(C_w, \delta_{\max})$ -convex shape field, with  $\delta_{\max} = 1$ .
- Any  $P \in \Gamma(x, M)$  satisfies  $|\partial^\beta P(x)| \leq M$  for  $|\beta| \leq m - 1$ .

We have the following corollary to Theorem 6.

**Corollary 4** *Let  $Q_0$  be a cube of sidelength  $\delta_{Q_0} \leq \delta_{\max}$ , and let  $x_0 \in E \cap Q_0$ . Let  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  be a  $(C_w, \delta_{\max})$ -convex shape field, and for  $l \geq 1$ , let  $\vec{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0}$  be its  $l^{\text{th}}$  refinement. Let  $M_0 > 0$ , and let  $P_0 \in \Gamma_{l_*}(x_0, M_0)$ , where  $l_*$  is a large enough integer constant determined by  $m$  and  $n$ .*

*Then there exists  $F \in C^m(Q_0)$  such that*

- $|\partial^\alpha F(x)| \leq C_* M_0$  for  $x \in Q_0$ ,  $|\alpha| = m$ .
- $J_x(F) \in \Gamma_0(x, C_* M_0)$  for all  $x \in E \cap Q_0$ .
- $J_{x_0}(F) = P_0$ .

*Here,  $C_*$  depends only on  $m$ ,  $n$ ,  $C_w$ .*

**Remark** *The corollary strengthens Theorem 6, because we now obtain  $J_{x_0}(F) = P_0$  rather than the weaker assertion  $|\partial^\beta(F - P_0)| \leq C_* M_0 \delta_{Q_0}^{m-|\beta|}$  for  $|\beta| \leq m$ . The  $l_*$  here is much bigger than in the proof of Theorem 6; maybe one can do better than here.*

To prove the corollary, we use a simple “clustering lemma”, namely Lemma 3.4 in [12].

**Lemma 21 (Clustering Lemma)** *Let  $S \subset \mathbb{R}^n$ , with  $2 \leq \#(S) \leq k^\#$ . Then  $S$  can be partitioned into nonempty subsets  $S_0, S_1, \dots, S_{v_{\max}}$ , such that  $\#(S_v) < \#(S)$  for each  $v$ , and  $\text{dist}(S_v, S_\mu) \geq c \cdot \text{diam}(S)$ , for  $\mu \neq v$ , with  $c$  depending only on  $k^\#$ .*

We use Lemma 21 to prove the following lemma.

**Lemma 22** *Let  $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$  be a shape field. For  $l \geq 1$ , let  $\vec{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0}$  be the  $l^{\text{th}}$  refinement of  $\vec{\Gamma}_0$ . Let  $l_* \geq 1$ ,  $M_0 > 0$ ,  $x_0 \in E$ ,*

and let  $P_0 \in \Gamma_{l_*}(x_0, M_0)$ . Then for any  $S \subset E$  with  $x_0 \in S$  and  $\#(S) \leq l_*$ , there exists a Whitney field  $\vec{P}^S = (P^x)_{x \in S}$  such that  $P^x \in \Gamma_0(x, C_* M_0)$  for  $x \in S$ ;

$$|\partial^\beta (P^x - P^y)(x)| \leq C_* M_0 |x - y|^{m-\beta} \text{ for } x, y \in S, |\beta| \leq m-1;$$

and  $P^{x_0} = P_0$ . Here,  $C_*$  depends only on  $m, n, l_*$ .

**Proof of Lemma 22.** Throughout the proof of Lemma 22,  $C_*$  denotes a constant determined by  $m, n, l_*$ .

We use induction on  $l_*$ .

In the base case,  $l_* = 1$ , so  $S \equiv \{x_0\}$ , and we can take our Whitney field  $\vec{P}^S$  to consist of the single polynomial  $P_0$ .

For the induction step, we fix  $l_* \geq 2$ , and make the inductive assumption: Lemma 22 holds with  $(l_* - 1)$  in place of  $l_*$ . We will then prove Lemma 22 for the given  $l_*$ .

Let  $\vec{l}, M_0, x_0, P_0, S$  be as in the hypotheses of Lemma 22.

If  $\#(S) \leq l_* - 1$ , then the conclusion of Lemma 22 follows instantly from our inductive assumption. Suppose  $\#(S) = l_*$ .

Let  $S_0, S_1, \dots, S_{v_{\max}}$  be a partition of  $S$  as in Lemma 21. We may assume that the  $S_v$  are numbered so that  $x_0 \in S_0$ . For each  $v = 1, \dots, v_{\max}$ , we pick a point  $x_v \in S_v$ .

Recall that  $P_0 \in \Gamma_{l_*}(x_0, M_0)$ . By definition of the  $l^{\text{th}}$  refinement, for each  $v = 1, \dots, v_{\max}$ , there exists

$$(12) \quad P_v \in \Gamma_{l_*-1}(x_v, M_0) \text{ such that } |\partial^\beta (P_v - P_0)(x_0)| \leq M_0 |x_v - x_0|^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Fix such  $P_v$  for  $v = 1, \dots, v_{\max}$ ; and note that (12) holds also for  $v = 0$ .

For each  $v = 0, 1, \dots, v_{\max}$ , we have  $P_v \in \Gamma_{l_*-1}(x_v, M_0)$ ,  $x_v \in S_v \subset E$ , and  $\#(S_v) \leq l_* - 1$ .

Hence our inductive assumption produces a Whitney field  $\vec{P}^{S_v} = (P_v^x)_{x \in S_v}$  such that  $P_v^x \in \Gamma_0(x, C_* M_0)$  for  $x \in S_v$ ,  $|\partial^\beta (P_v^x - P_v^y)(x)| \leq C_* M_0 |x - y|^{m-|\beta|}$  for  $x, y \in S_v$ ,  $|\beta| \leq m-1$ ,  $P_v^{x_v} = P_v$ .

We now combine the  $\vec{P}^{S_v}$  into a single Whitney field  $\vec{P}^S = (P^x)_{x \in S}$  by setting  $P^x = P_v^x$  for  $x \in S_v$ . Note that  $P^x \in \Gamma_0(x, C_* M_0)$  for  $x \in S$ , and that  $P^{x_0} = P_0^{x_0} = P_0$ . We check that

$$(13) \quad |\partial^\beta (P^x - P^y)(x)| \leq C_* M_0 |x - y|^{m-|\beta|} \text{ for } x, y \in S, |\beta| \leq m-1.$$

We already know (13) for  $x, y$  belonging to the same  $S_v$ .

Suppose  $x \in S_v, y \in S_\mu$  with  $\mu \neq v$ . Then  $|x - y|$  is comparable to  $\text{diam}(S)$  so (13) asserts that

$$(14) \quad |\partial^\beta (P^x - P^y)(x)| \leq C_* M_0 (\text{diam}(S))^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

We know that

$$(15) \quad |\partial^\beta (P^x - P_v)(x_v)| = |\partial^\beta (P_v^x - P_v^{x_v})(x_v)| \leq C_* M_0 |x - x_v|^{m-|\beta|} \leq C_* M_0 (\text{diam}(S))^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Similarly,

$$(16) \quad |\partial^\beta (P^y - P_\mu)(x_\mu)| \leq C_* M_0 (\text{diam}(S))^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

$$(17) \quad \text{Also, } |\partial^\beta (P_v - P_0)(x_0)| \leq C_* M_0 (\text{diam}(S))^{m-|\beta|} \text{ for } |\beta| \leq m-1, \text{ and}$$

$$(18) \quad |\partial^\beta (P_\mu - P_0)(x_0)| \leq C_* M_0 (\text{diam}(S))^{m-|\beta|} \text{ for } |\beta| \leq m-1.$$

Because the points  $x_0, x_\mu, x_v, x$  all lie in  $S$ , the distance between any two of these points is at most  $\text{diam}(S)$ .

Hence, the estimates (15),  $\dots$ , (18) together imply (14), completing the proof of (13). This completes our induction on  $l_*$ , thus establishing Lemma 22. ■

**Proof of Corollary 4.** We set  $\hat{\Gamma} = (\hat{\Gamma}(x, M))_{x \in E, M > 0}$ , where

$$\hat{\Gamma}(x, M) = \begin{cases} \Gamma_0(x, M) & \text{if } x \in E \setminus \{x_0\} \\ \{P_0\} & \text{if } x = x_0 \end{cases}.$$

One checks trivially that  $\hat{\Gamma}$  is a  $(C_w, \delta_{\max})$ -convex shape field.

By applying Lemma 22 with  $l_* = k^\# + 1$  (so that if necessary we can add  $x_0$  into the set  $S$  below), we obtain the following conclusion:

$$(19) \quad \text{Given } S \subset E \text{ with } \#(S) \leq k^\#, \text{ there exists a Whitney field } \vec{P}^S = (P^x)_{x \in S} \text{ such that } P^x \in \hat{\Gamma}(x, C_* M_0) \text{ for } x \in S, \text{ and } |\partial^\beta (P^x - P^y)(x)| \leq C_* M_0 |x - y|^{m-|\beta|} \text{ for } x, y \in S, |\beta| \leq m-1.$$

Here,  $C_*$  depends only on  $m, n, k^\#$ .

For large enough  $k^\#$  depending only on  $m, n$ , (19) and Theorem 6 together imply the conclusion of our corollary.

The proof of Corollary 4 is complete. ■

## III.2 Finiteness Principle II

**Proof of Theorem 3 (A).** Let us first set up notation. We write  $c, C, C'$ , etc., to denote constants determined by  $m, n, D$ ; these symbols may denote different constants in different occurrences. We will work with  $C^m$  vector and scalar-valued functions on  $\mathbb{R}^n$ , and also with  $C^{m+1}$  scalar-valued functions on  $\mathbb{R}^{n+D}$ . We use Roman letters  $(x, y, z, \dots)$  to denote points of  $\mathbb{R}^n$ , and Greek letters  $(\xi, \eta, \zeta, \dots)$  to denote points of  $\mathbb{R}^D$ . We denote points of the  $\mathbb{R}^{n+D}$  by  $(x, \xi), (y, \eta)$ , etc. As usual,  $\mathcal{P}$  denotes the vector space of real-valued polynomials of degree at most  $m-1$  on  $\mathbb{R}^n$ . We write  $\mathcal{P}^D$  to denote the direct sum of  $D$  copies of  $\mathcal{P}$ . If  $F \in C^{m-1}(\mathbb{R}^n, \mathbb{R}^D)$  with  $F(x) = (F_1(x), \dots, F_D(x))$  for  $x \in \mathbb{R}^n$ , then  $J_x(F) := (J_x(F_1), \dots, J_x(F_D)) \in \mathcal{P}^D$ .



We write  $\mathcal{P}^+$  to denote the vector space of real-valued polynomials of degree at most  $m$  on  $\mathbb{R}^{n+D}$ . If  $F \in C^{m+1}(\mathbb{R}^{n+D})$ , then we write  $J_{(x,\xi)}^+ F \in \mathcal{P}^+$  to denote the  $m^{\text{th}}$ -degree Taylor polynomial of  $F$  at the point  $(x, \xi) \in \mathbb{R}^{n+D}$ .

When we work with  $\mathcal{P}^+$ , we write  $\odot_{(x,\xi)}$  to denote the multiplication

$$P \odot_{(x,\xi)} Q := J_{(x,\xi)}^+ (PQ) \in \mathcal{P}^+ \text{ for } P, Q \in \mathcal{P}^+.$$

We will use Theorem 6 for  $C^{m+1}$ -functions on  $\mathbb{R}^{n+D}$ . (Compare with [17].) Thus,  $m+1$  and  $n+D$  will play the rôles of  $m, n$ , respectively, when we apply that theorem.

We take  $k^\#$  as in Theorem 6, where we use  $m+1, n+D$  in place of  $m, n$ , respectively.

We now introduce the relevant shape field.

Let  $E^+ = \{(x, 0) \in \mathbb{R}^{n+D} : x \in E\}$ . For  $(x_0, 0) \in E^+$  and  $M > 0$ , let

$$(1) \quad \Gamma((x_0, 0), M) = \left\{ P \in \mathcal{P}^+ : P(x_0, 0) = 0, \nabla_\xi P(x_0, 0) \in K(x_0), \right. \\ \left. \left| \partial_x^\alpha \partial_\xi^\beta P(x_0, 0) \right| \leq M \text{ for } |\alpha| + |\beta| \leq m \right\} \subset \mathcal{P}^+.$$

$$\text{Let } \vec{\Gamma} = (\Gamma(x_0, 0), M)_{(x_0, 0) \in E^+, M > 0}.$$

**Lemma 23**  $\vec{\Gamma}$  is a  $(C, 1)$ -convex shape field.

The proof of the lemma follows easily from the following observation: If  $P_1, P_2 \in \Gamma((x_0, 0), M)$  and  $Q_1, Q_2 \in \mathcal{P}^+$ , then  $P_1(x_0, 0) = P_2(x_0, 0) = 0$ , hence  $P := Q_1 \odot_{(x_0, 0)} Q_1 \odot_{(x_0, 0)} P_1 + Q_2 \odot_{(x_0, 0)} Q_2 \odot_{(x_0, 0)} P_2$  satisfies  $P(x_0, 0) = 0$  and  $\nabla_\xi P(x_0, 0) = (Q_1(x_0, 0))^2 \nabla_\xi P_1(x_0, 0) + (Q_2(x_0, 0))^2 \nabla_\xi P_2(x_0, 0)$ .

**Lemma 24** Let  $S^+ \subset E^+$  with  $\#(S^+) \leq k^\#$ . Then there exists  $\vec{P} = (P^z)_{z \in S^+}$ , with each  $P^z \in \mathcal{P}^+$ , such that

$$(2) \quad P^z \in \Gamma(z, C) \text{ for each } z \in S^+, \text{ and}$$

$$(3) \quad \left| \partial_x^\alpha \partial_\xi^\beta (P^z - P^{z'}) (z) \right| \leq C |z - z'|^{(m+1)-|\alpha|-|\beta|} \text{ for } z, z' \in S^+ \text{ and } |\alpha| + |\beta| \leq m.$$

**Proof of Lemma 24.** Since  $E^+ = E \times \{0\}$ , we have  $S^+ = S \times \{0\}$  for an  $S \subset E$  with  $\#(S) \leq k^\#$ . By hypothesis of Theorem 3 (A), there exists  $F^S \in C^m(\mathbb{R}^n, \mathbb{R}^D)$  such that

$$(4) \quad \|F^S\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq 1 \text{ and } F^S(x_0) \in K(x_0) \text{ for all } x_0 \in S.$$

Let  $F^S(x) = (F_1^S(x), \dots, F_D^S(x))$  for  $x \in \mathbb{R}^n$ , and let  $\vec{P} = (P^{(x_0, 0)})_{(x_0, 0) \in S \times \{0\}}$  with

$$(5) \quad P^{(x_0, 0)}(x, \xi) = \sum_{i=1}^D \xi_i [J_{x_0}(F_i^S)(x)] \text{ for } x \in \mathbb{R}^n, \xi = (\xi_1, \dots, \xi_D) \in \mathbb{R}^D.$$

Each  $P^{(x_0,0)}$  belongs to  $\mathcal{P}^+$  and satisfies

$$(6) \quad P^{(x_0,0)}(x_0,0) = 0, \nabla_\xi P^{(x_0,0)}(x_0,0) \in K(x_0), \text{ and}$$

and

$$(7) \quad \left| \partial_x^\alpha \partial_\xi^\beta P^{(x_0,0)}(x_0,0) \right| \leq C \text{ for } |\alpha| + |\beta| \leq m,$$

thanks to (4), (5). Our results (6), (7) and definition (1) together imply (2). We pass to (3). Let  $(x_0,0), (y_0,0) \in S^+ = S \times \{0\}$ . From (4), (5), we have

$$\begin{aligned} \left| \partial_x^\alpha \partial_{\xi_j} \left( P^{(x_0,0)} - P^{(y_0,0)} \right) (x_0,0) \right| &= \left| \partial_x^\alpha (J_{x_0}(F_j^S) - J_{y_0}(F_j^S))(x_0) \right| \\ &\leq C |x_0 - y_0|^{m-|\alpha|} \\ &= C |x_0 - y_0|^{(m+1)-(|\alpha|+1)} \end{aligned}$$

for  $|\alpha| \leq m-1, j = 1, \dots, D$ . For  $|\beta| \neq 1$ , we have

$$\partial_x^\alpha \partial_\xi^\beta \left( P^{(x_0,0)} - P^{(y_0,0)} \right) (x_0,0) = 0$$

by (5). The above remarks imply (3), completing the proof of Lemma 24.  $\blacksquare$

**Lemma 25** *Given a cube  $Q \subset \mathbb{R}^n$  of sidelength  $\delta_Q = 1$ , there exists  $F^Q \in C^m(Q, \mathbb{R}^D)$  such that*

$$(8) \quad \left| \partial^\alpha F^Q(x) \right| \leq C \text{ for } x \in Q, |\alpha| \leq m; \text{ and}$$

$$(9) \quad F^Q(z) \in K(z) \text{ for all } z \in E \cap Q.$$

**Proof of Lemma 25.** If  $E \cap Q = \emptyset$ , then we can just take  $F^Q \equiv 0$ . Otherwise, pick  $x_{00} \in E \cap Q$ , let  $Q' \in \mathbb{R}^D$  be a cube of sidelength  $\delta_{Q'} = 1$ , containing the origin in its interior, and apply Theorem 6 (with  $m+1, n+D$  in place of  $m, n$ , respectively) to the shape field  $\vec{\Gamma} = (\Gamma(x_0,0), M)_{(x_0,0) \in E^+, M > 0}$  given by (1), the cube  $Q_0 := Q \times Q' \subset \mathbb{R}^{n+D}$ , the point  $(x_{00},0)$ , and the number  $M_0 = C$ .

Lemmas 23 and 24 tell us that the above data satisfy the hypotheses of Theorem 6. Applying Theorem 6, we obtain

$$(10) \quad P_0 \in \Gamma((x_{00},0), C) \text{ and } F \in C^{m+1}(Q \times Q') \text{ such that}$$

$$(11) \quad \left| \partial_x^\alpha \partial_\xi^\beta (F - P_0)(x, \xi) \right| \leq C \text{ for } |\alpha| + |\beta| \leq m+1 \text{ and } (x, \xi) \in Q \times Q'; \text{ and}$$

$$(12) \quad J_{(z,0)}^+(F) \in \Gamma((z,0), C) \text{ for all } z \in E \cap Q.$$

By (10), (12) and definition (1), we have

$$(13) \quad \left| \partial_x^\alpha \partial_\xi^\beta P_0(x_{00},0) \right| \leq C \text{ for } |\alpha| + |\beta| \leq m$$

and

$$(14) \quad \nabla_{\xi} F(z, 0) \in K(z) \text{ for all } z \in E \cap Q.$$

Since  $(x_{00}, 0) \in Q \times Q'$  and  $\delta_{Q \times Q'} = 1$ , (13) implies that

$$\left| \partial_x^{\alpha} \partial_{\xi}^{\beta} P_0(x, \xi) \right| \leq C$$

for  $(x, \xi) \in Q \times Q'$ ,  $|\alpha| + |\beta| \leq m + 1$ . (Recall that  $P_0$  is a polynomial of degree at most  $m$ .) Together with (11), this implies that

$$(15) \quad \left| \partial_x^{\alpha} \partial_{\xi}^{\beta} F(x, \xi) \right| \leq C \text{ for } (x, \xi) \in Q \times Q', |\alpha| + |\beta| \leq m + 1.$$

Taking

$$F^Q(x) = \nabla_{\xi} F(x, 0) \text{ for } x \in Q,$$

we learn from (14), (15) that  $F^Q \in C^m(Q, \mathbb{R}^D)$ ;  $|\partial^{\alpha} F^Q(x)| \leq C$  for  $x \in Q$ ,  $|\alpha| \leq m$ ; and  $F^Q(z) \in K(z)$  for all  $z \in E \cap Q$ . Thus,  $F^Q$  satisfies (8) and (9), completing the proof of Lemma 25. ■

It is now trivial to complete the proof of Theorem 3 (A) by using a partition of unity on  $\mathbb{R}^n$ . ■

**Proof of Theorem 3 (B).** We write  $c$ ,  $C$ ,  $C'$ , etc., to denote constants determined by  $m$ ,  $n$ ,  $D$ . These symbols may denote different constants in different occurrences.

Suppose first that  $E$  is finite.

Given  $F^S$  as in the hypothesis of Theorem 3 (B), set  $\vec{P} = (P^x)_{x \in S}$ , with  $P^x = J_x(F^S) \in \mathcal{P}^D$ . Then  $P^x(x) \in K(x)$  for  $x \in S$ ,  $|\partial^{\beta} P^x(x)| \leq C$  for  $x \in S$ ,  $|\beta| \leq m - 1$ , and  $|\partial^{\beta} (P^x - P^y)(x)| \leq C|x - y|^{m-|\beta|}$  for  $x, y \in S$ ,  $|\beta| \leq m - 1$ .

By Whitney's extension theorem for finite sets, there exists

- $\tilde{F}^S \in C^m(\mathbb{R}^n, \mathbb{R}^D)$  such that

- $\left\| \tilde{F}^S \right\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq C$ , and

$J_x(\tilde{F}^S) = P^x$  for  $x \in S$ ; in particular,

- $\tilde{F}^S(x) = F^S(x) \in K(x)$  for  $x \in S$ .

Thanks to the above bullet points, we have satisfied the hypotheses of Theorem 3 (A). Hence, we obtain  $F \in C^m(\mathbb{R}^n, \mathbb{R}^D) \subset C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$  such that  $\|F\|_{C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)} \leq C \|F\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)} \leq C'$ , and  $F(x) \in K(x)$  for each  $x \in E$ . Thus, we have proven Theorem 3 (B) in the case of finite  $E$ .

Next, suppose  $E$  is an arbitrary subset of a cube  $Q \subset \mathbb{R}^n$ . Then

$$X = \left\{ F \in C^{m-1,1}(Q, \mathbb{R}^D) : \|F\|_{C^{m-1,1}(Q, \mathbb{R}^D)} \leq C \right\}$$

is compact in the topology of the  $C^{m-1}(Q, \mathbb{R}^D)$ -norm, by Ascoli's theorem.

For each  $x \in E$ , let

$$X(x) = \{F \in X : F(x) \in K(x)\}.$$

Then each  $X(x)$  is a closed subset of  $X$ , since  $K(x) \subset \mathbb{R}^n$  is closed. Moreover, given finitely many points  $x_1, \dots, x_N \in E$ , we have  $X(x_1) \cap \dots \cap X(x_N) \neq \emptyset$ , thanks to Theorem 3 (B) in the known case of finite sets.

Consequently,  $\bigcap_{x \in E} X(x) \neq \emptyset$ . Thus, there exists  $F \in C^{m-1,1}(Q, \mathbb{R}^D)$  such that

$$(16) \quad \|F\|_{C^{m-1,1}(Q, \mathbb{R}^D)} \leq C \text{ and } F(x) \in K(x) \text{ for all } x \in E.$$

We have achieved (16) under the assumption  $E \subset Q$ .

Finally, Theorem 3 (B) for arbitrary  $E \subset \mathbb{R}^n$  follows from the known case  $E \subset Q$  by an obvious argument using a partition of unity. ■

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